SÁRKÖZY'S THEOREM FOR SUMS OF AN EVEN NUMBER OF ODD POWERS

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Abstract. We obtain a Sárközy-type result for sets $A \subseteq [N]$ whose difference set $A - A$ does not contain the sum of an even number of k-th powers of positive integers, with k odd. Namely, we prove that such sets must satisfy a power-saving bound $|A| \ll_{k,\varepsilon} N^{1-\tau_k+\varepsilon}$ for any fixed $\varepsilon > 0$, where we can take $\tau_k = \max\left\{\frac{2^{1-k}}{k}, \frac{1}{k^2(k-1)}\right\}$ using the classical theory and the best currently available bounds for classical Weyl sums.

1. Introduction

Let $k > 1$ be a fixed odd integer, s be a fixed even positive integer, and let

$$\mathcal{A}_{k,s} = \{a_1^k + \cdots + a_s^k : \ a_1, \ldots, a_s \in \mathbb{N}\}.$$

We study sets $A \subseteq [N]$ whose difference set $A - A$ contains no element of $\mathcal{A}_{k,s}$. Our purpose is to obtain a power saving bound of the form $|A| \ll_k N^{1-\tau_k}$ for some $\tau_k > 0$. The exact value of the constant τ_k is dependent on the admissible exponent $\sigma_k > 0$ in the classical Weyl estimate

$$\sum_{m \leq X} e(\alpha m^k) \ll_{k,\delta} X^{1+\delta} \left(\frac{1}{q} + \frac{1}{X} + \frac{q}{X^k}\right)^{\sigma_k}, \qquad (e(\alpha) = e^{2\pi i \alpha}) \tag{1.1}$$

whenever $|\alpha - a/q| \leq q^{-2}$. Note that any exponent admissible in (1.1) necessarily satisfies $\sigma_k \leq 1/k$: Indeed, for any prime $p \nmid k$, taking $X = q = p^k$ and $\alpha = a/p^k$ gives

$$\sum_{m \leq X} e(\alpha m^k) = p^{k-1} = X^{1-1/k}.$$

Define $\tau_k = \sigma_k / k$. Our main result is the following.

Theorem 1.1. *Fix any positive even integer s, and any odd integer $k > 1$. For all $N \in \mathbb{N}$ and for any set $A \subseteq [N]$ satisfying $(A - A) \cap \mathcal{A}_{k,s} = \emptyset$, we have*

$$|A| \ll_{k,s,\epsilon} N^{1-\tau_k+\epsilon} \text{ for any fixed } \epsilon > 0.$$

Theorem 1.1 is only non-trivial for all N sufficiently large in terms of k and ϵ. It might be interesting to note that the bound we get is independent of s; of course, the bound is nontrivial only for s smaller than the Waring number (i.e., the least threshold r such that all sufficiently large positive integers can be written as a sum of r many positive k-th powers). The classical Weyl estimate gives $\sigma_k = 2^{1-k}$ (see chapter 2 of [?]), while the best currently available bound (for $k > 5$) is $\sigma_k = \frac{1}{k(k-1)}$ [?], thus one can take $\tau_k = \frac{1}{k^2(k-1)}$. In particular,

2020 *Mathematics Subject Classification.* Primary 11A25; Secondary 11N36, 11N37, 11N64, 11N69.
Key words and phrases. Sárközy's theorem, van der Corput property, circle method, More keywords?

for $k = 3$ and 5, we can take $\sigma_k = 1/4$ and $\sigma_k = 1/16$, which by Theorem 1.1 leads to the bounds $|A| \ll_\varepsilon N^{11/12+\varepsilon}$ and $|A| \ll_\varepsilon N^{79/80+\varepsilon}$ respectively; likewise for odd $k > 5$, Theorem 1.1 yields $|A| \ll_{k,\varepsilon} N^{1-1/(k^2(k-1))+\varepsilon}$.

Two immediate consequences of Theorem 1.1 that are worth point out are as follows: First, is the simplest special case, the "forbidden difference problem" for avoiding sums of two k-th powers, which we include not only because it is morally the strongest case, but also to make it easier for the reader to parse our results.

Corollary 1.2. *Fix any odd integer $k > 1$. For all $N \in \mathbb{N}$ and for any set $A \subseteq [N]$ satisfying $(A - A) \cap \{a^k + b^k : a, b \in \mathbb{N}\} = \emptyset$ we have,*

$$|A| \ll_{k,\epsilon} N^{1-\tau_k+\epsilon} \text{ for any fixed } \epsilon > 0.$$

The second consequence we highlight is the problem of avoiding sums of **non–negative** k-th powers. Note that if $A - A$ avoids the set $\{a_1^k + \cdots + a_s^k : a_1, \ldots, a_s \in \mathbb{Z}_{\geq 0}\} \setminus \{0\}$ for **any** integer $s > 1$ (*irrespective* of its parity), then $A - A$ must also avoid the set $\{a^k + b^k : a, b \in \mathbb{N}\}$. Hence, the following is actually a consequence of the previous corollary.

Corollary 1.3. *Fix any odd integer $k > 1$ and **any** integer $s > 1$. For all $N \in \mathbb{N}$ and for any set $A \subseteq [N]$ satisfying $(A - A) \cap \{a_1^k + \cdots + a_s^k : a_1, \ldots, a_s \in \mathbb{Z}_{\geq 0}\} \subseteq \{0\}$ we have,*

$$|A| \ll_{k,\epsilon} N^{1-\tau_k+\epsilon} \text{ for any fixed } \epsilon > 0.$$

For any two fixed odd positive integers $k_1, \ldots, k_s$, note that if a set avoids all integers of the form $a_1^{k_1} + \cdots + a_s^{k_s}$ with $a_1, \ldots, a_s \in \mathbb{N}$, then it also avoids integers of the form $a_1^K + \cdots + a_s^K$, where $K := [k_1, \ldots, k_s]$ is the least common multiple of $k_1, \ldots, k_s$. Hence, an immediate generalization of Theorem 1.1 is the following.

Theorem 1.4. *Fix any positive even integer s, any positive odd integers $k_1, \ldots, k_s$, at least one of which is not 1, and let $K := [k_1, \ldots, k_s]$. Then for any set $A \subseteq [N]$ such that its difference set $A - A$ does not contain any integer of the form $a_1^{k_1} + \cdots + a_s^{k_s}$, with $a_1, \ldots, a_s \in \mathbb{N}$, we have*

$$|A| \ll_{k_1,\ldots,k_s,\epsilon} N^{1-\tau_K+\epsilon} \text{ for any fixed } \epsilon > 0.$$

Let us describe our construction. Define

$$r_{k,s}(n) = \#\{(a_1, \ldots, a_s) \in \mathbb{N}^s : \ a_1^k + \cdots + a_s^k = n\}.$$

Since $\sum_{n \leq N} r_{k,s}(n) \asymp N^{s/k}$, the density of $r_{k,s}(n)$ at scale n is roughly $n^{s/k-1}$. This suggests compensating for the sparsity of the sums of two k-th powers by using the weight

$$\nu_N(d) := r_{k,s}(|d|)|d|^{1-s/k}\left(1 - \frac{|d|}{N}\right)_+ = \mathbb{1}_{d \in (-N,N)}\, r_{k,s}(|d|)\, |d|^{1-s/k}\, \left(1 - \frac{|d|}{N}\right)$$

for $d \neq 0$ and $\nu_N(0) := 0$. Note that the function $\nu_N(d)$ is supported exactly on those integers in $d \in (-N, N)$ for which $|d|$ can be written in the form $a_1^k + \cdots + a_s^k$ for some $a, b \in \mathbb{N}$. The kernel $\left(1 - \frac{|d|}{N}\right)$ is inspired by Green [**?**]. As remarked at the end of the manuscript, our entire argument goes through if the exponent of $|d|$ in the definition of $\nu_N(d)$ were taken to be any constant $c \in (-s/k, 1-s/k]$; moreover, every such choice of c gives the same result, namely Theorem 1.1.

Our arguments can be summarized as follows. To begin with, note that if $(A-A)\cap \mathcal{A}_{k,s} = \emptyset$, then

(1.2)
$$\int_0^1 |\widehat{\mathbb{1}}_A(\alpha)|^2\, \widehat{\nu}_N(\alpha)\, d\alpha \;=\; \sum_{\substack{m,n\in A\\ -N<r<N}} \nu_N(r) \int_0^1 e((n-m+r)\alpha)\, d\alpha \;=\; \sum_{m,n\in A} \nu_N(m-n) \;=\; 0.$$

Denoting by $\hat{f} : \mathbb{T} \to \mathbb{C}$ the Fourier transform of a function $f : \mathbb{Z} \to \mathbb{C}$, defined as

$$\hat{f}(\alpha) = \sum_{n\in\mathbb{Z}} f(n) e(n\alpha), \tag{1.3}$$

and taking into account $(A-A)\cap \mathcal{A}_{k,s} = \emptyset$, we find

(1.4)
$$0 = \sum_{m,n\in A} \nu_N(m-n) \;=\; \sum_{\substack{m,n\in A\\ -N<r<N}} \nu_N(r) \int_0^1 e((n-m+r)\alpha)\, d\alpha = \int_0^1 |\widehat{\mathbb{1}}_A(\alpha)|^2\, \widehat{\nu}_N(\alpha)\, d\alpha$$

The main feature of the weight ν_N is revealed by its major arcs approximation. If

$$\alpha = \frac{a}{q} + \beta$$

and

$$S_k(q,a) = \sum_{r \pmod q} e(ar^k/q),$$

then an extension of an estimate of Brüdern and Vaughan [?] (see Lemmas 2.2 and 2.3 below) gives

$$\widehat{\nu}_N(\alpha) = 2C_{k,s}\left(\frac{S_k(q,a)}{q}\right)^s \int_0^N \cos(2\pi\beta w)\, \left(1-\frac{w}{N}\right)\, dw \;+\; \text{error} \tag{1.5}$$

for some constant $C_{k,s} > 0$. Both factors in the main term above are non-negative: Indeed,

$$2\int_0^N \cos(2\pi\beta w)\, \left(1-\frac{w}{N}\right)\, dw = N\left(\frac{\sin(N\pi\beta)}{\pi N\beta}\right)^2 \geq 0, \tag{1.6}$$

while the odd parity of k and the even parity of s imply that $S_k(q,a)^s \in \mathbb{R}^+$, since

$$\overline{S_k(q,a)} = \sum_{r \bmod q} e\left(-\frac{ar^k}{q}\right) = \sum_{r \bmod q} e\left(\frac{a(-r)^k}{q}\right) = \sum_{m \bmod q} e\left(\frac{am^k}{q}\right) = S_k(q,a). \tag{1.7}$$

Hence the contribution of every major arc to the counting function $\int_0^1 |\widehat{\mathbb{1}}_A(\alpha)|^2\, \widehat{\nu}_N(\alpha)\, d\alpha$ is not too negative; in fact, the total contribution of the major arcs turns out to be $O(|A|N^{1-0.5/k+\varepsilon})$. Moreover the contribution of the major arc around 0 is $\approx |A|^2$. Using an extension (Lemma 2.4) of the classical Weyl estimate (1.1), the contributions of the minor arcs is $O(|A|N^{1-\sigma_k/k+\varepsilon})$. Combining all these estimates with (1.4), we obtain the desired bound $|A| \ll N^{1-\tau_k+\varepsilon}$.

The positivity of the main term of (1.5) is the main reason the case of odd k admits a relatively direct argument. For a single odd polynomial, the analogous complete sum is real but may be negative, and previous positive-exponential-sum constructions of Ninčevič–Slijepčevič [?] average over several dilations to overcome this problem. Fortunately, for sums of an even number of equal odd powers, the complete sum $S_k(q,a)$ is automatically raised to an even power, thus yielding the desired positivity. By contrast, for sums of two squares the quadratic complete sums may be negative or complex; the recent work of Fan and Lott [?] resolves this by averaging over dilations in the style of Ninčevič–Slijepčevič and then carefully choosing smooth weights which force the relevant complete sum and oscillatory integral to lie in a fixed sector thus yielding the positivity of the real part of their product.

Our approach is equivalent to the quantitative van der Corput method. This idea goes back to Kamae-Mendés France [?] and Ruzsa [?]; see also Montgomery [?] and the general framework of Matolcsi and Ruzsa [?] which is then applied to cubic residues in [?]. A recent breakthrough due to Green [?] applies this approach to obtain a power saving bound for sets whose different set does not contain a shifted prime $p-1$. Although we formulate the proof directly, our argument in fact constructs a power-saving van der Corput witness function for the set $\mathcal{A}_{k,s}$.

More precisely, with $\mathcal{A}_{k,s}(N) \coloneqq \mathcal{A}_{k,s} \cap [N]$, our proof produces non-negative coefficients c_d, supported on $\mathcal{A}_{k,s}(N)$, satisfying

$$\sum_{d\in\mathcal{A}_{k,s}(N)} c_d = 1 \quad\text{and}\quad \operatorname{Re}\sum_{d\in\mathcal{A}_{k,s}(N)} c_d\, e(d\alpha) \;\geq\; -C_{k,s,\varepsilon}\, N^{-\tau_k+\varepsilon} \qquad (\alpha\in\mathbb{T}).$$

Theorem 1.1 then follows by a short, standard Fourier analytic argument.

Notations and conventions. In the rest of the manuscript $[a,b]$ just denotes the closed interval, with left endpoint a and right endpoint b. Implied constants in $\ll$ and O-notation, and implicit constants in qualifiers like "sufficiently large", may depend on any parameters declared as "fixed". In particular, all implied constants are always allowed to depend on the fixed parameters k, s and ϵ.

2. The key major and minor arc estimates

The first input in our arguments will be the following restated version of Theorem 4.1 in [?].

Theorem 2.1. *Fix any $k\in\mathbb{N}_{>1}$ and $\epsilon>0$, and define $S_k(q,a) \coloneqq \sum_{r \bmod q} e(ar^k/q)$. Then*

$$\begin{aligned}\sum_{n\leq X} e\left(\alpha n^k\right) \;&=\; \frac{S_k(q,a)}{q}\int_0^X e(\beta t^k)\,dt + O(q^{1/2+\epsilon}(1+X^k|\beta|)^{1/2})\\ &=\; \frac{S_k(q,a)}{kq}\int_0^{X^k}\frac{e(\beta u)}{u^{1-1/k}}\,du \,+\, O(q^{1/2+\epsilon}(1+X^k|\beta|)^{1/2}),\end{aligned}$$

uniformly in $X\geq 1$, in $q\in\mathbb{N}$, in $a\in\mathbb{Z}$, and in $\beta\in\mathbb{R}$, where $\alpha \coloneqq a/q+\beta$.

We start by establishing the following extension of Lemma 2.5 in [?], which gives an explicit estimate on the exponential sum $\sum_{n\leq N} r_{k,s}(n)e(n\alpha)$, uniformly in all real phases α.

Lemma 2.2. *Fix any $k, s \in \mathbb{N}_{>1}$ and $\epsilon > 0$, and define $C_{k,s} := \Gamma(1+1/k)^s/\Gamma(s/k) > 0$. We have*

$$\sum_{n\le N} r_{k,s}(n)e(n\alpha) = C_{k,s}\left(\frac{S_k(q,a)}{q}\right)^s \int_0^N \frac{e(\beta w)}{w^{1-s/k}}\,dw \;+\; O(N^{(s-1)/k}q^{1/2+\epsilon}(1+N|\beta|o)^{1/2}),$$

uniformly in all real β, and in integers $N \ge 1$, $q \ge 1$ and a, where $\alpha = a/q+\beta$.

Proof. We will establish the lemma by inducting on n. In the base case $s = 2$, we mostly follow part of the argument of [**?**, Lemma 2.5], with some changes. We start by writing

$$\sum_{n\le N} r_{k,2}(n)e(n\alpha) \;=\; \sum_{\substack{x,y\ge 1\\ x^k+y^k\le N}} e(\alpha(x^k+y^k)) \;=\; \sum_{1\le x\le N^{1/k}} e(\alpha x^k) \sum_{1\le y\le (N-x^k)^{1/k}} e(\alpha y^k),$$

and then using Theorem 2.1 to get

(2.1)
$$\begin{aligned}\sum_{n\le N} r_{k,2}(n)e(n\alpha) \;&=\; \sum_{1\le x\le N^{1/k}} e(\alpha x^k)\left\{\frac{S_k(q,a)}{kq}\int_0^{N-x^k}\frac{e(\beta u)}{u^{1-1/k}}\,du \;+\; O(q^{1/2+\epsilon}(1+(N-x^k)|\beta|)^{1/2})\right\}\\ &=\; \frac{S_k(q,a)}{kq}\sum_{1\le x\le N^{1/k}} e(\alpha x^k)\int_0^{N-x^k}\frac{e(\beta u)}{u^{1-1/k}}\,du \;+\; O(N^{1/k}q^{1/2+\epsilon}(1+N|\beta|)^{1/2}).\end{aligned}$$

Interchanging the sum and the integral, we see that the main term above is

$$\frac{S_k(q,a)}{kq}\int_0^N \frac{e(\beta u)}{u^{1-1/k}}\cdot\left(\sum_{x\le (N-u)^{1/k}} e(\alpha x^k)\right)du,$$

and another application of Theorem 2.1 shows that this equals

$$\begin{aligned}&\frac{S_k(q,a)}{kq}\int_0^N \frac{e(\beta u)}{u^{1-1/k}}\cdot\left\{\frac{S_k(q,a)}{kq}\int_0^{N-u}\frac{e(\beta v)}{v^{1-1/k}}\,dv \;+\; O(q^{1/2+\epsilon}(1+(N-u)|\beta|)^{1/2})\right\}du\\ &=\left(\frac{S_k(q,a)}{kq}\right)^2 I_{k,2}(\beta,N) \;+\; O\left(q^{1/2+\epsilon}(1+N|\beta|)^{1/2}\int_0^N u^{1/k-1}\,du\right)\end{aligned}$$

(2.2)
$$=\left(\frac{S_k(q,a)}{q}\right)^2\cdot\frac{I_{k,2}(\beta,N)}{k^2} \;+\; O\left(N^{1/k}q^{1/2+\epsilon}(1+N|\beta|)^{1/2}\right),$$

where $I_{k,2}(\beta,N) := \int_0^N u^{1/k-1}e(\beta u)\int_0^{N-u} v^{1/k-1}e(\beta v)\,dv\,du$, and in the second line of the above computations, we have used the trivial bound $|S_k(q,a)| \le q$. Inserting (2.2) into (2.1), we obtain

$$\sum_{n\le N} r_{k,2}(n)e(n\alpha) \;=\; \left(\frac{S_k(q,a)}{q}\right)^2\cdot\frac{I_{k,2}(\beta,N)}{k^2} \;+\; O\left(N^{1/k}q^{1/2+\epsilon}(1+N|\beta|)^{1/2}\right).$$

Since $\Gamma(1+1/k) = k^{-1}\,\Gamma(1/k)$, the base case $s=2$ of the lemma will be proved once we show that

$$I_{k,2}(\beta,N) \;=\; \frac{\Gamma(1/k)^2}{\Gamma(2/k)} \int_0^N w^{2/k-1}\, e(\beta w)\, dw. \tag{2.3}$$

To show this, we start by letting $w := u+v$ to see that

$$I_{k,2}(\beta,N) = \int_0^N u^{1/k-1} \int_0^{N-u} v^{1/k-1} e(\beta(u+v))\, dv\, du \;=\; \int_0^N u^{1/k-1} \int_u^N (w-u)^{1/k-1} e(\beta w)\, dw\, du.$$

Interchanging the two integrals in the last expression and then taking $u =: wz$, we find that

$$\begin{aligned} I_{k,2}(\beta,N) &= \int_0^N e(\beta w) \int_0^w u^{1/k-1}\,(w-u)^{1/k-1}\, du\, dw \\ &= \int_0^N w^{2/k-1}\, e(\beta w) \int_0^1 z^{1/k-1}\,(1-z)^{1/k-1}\, dz\, dw \end{aligned}$$

The identity (2.3) now follows from the fact that the integral $\int_0^1 z^{1/k-1}\,(1-z)^{1/k-1}\,dz$ is the beta function $B(1/k,1/k) = \Gamma(1/k)^2/\Gamma(2/k)$. This completes the proof of the base case $s=2$.

Assume now that Lemma 2.2 holds for some $s \geq 2$; we will show that it holds for $s+1$. The inductive step is mostly analogous to the case $s=2$, so we just mention the main changes. Writing

$$\begin{aligned} \sum_{n\leq N} r_{k,s+1}(n)\, e(n\alpha) \;&=\; \sum_{1\leq x_{s+1}\leq N^{1/k}} e(\alpha x_{s+1}^k) \sum_{\substack{x_1,\ldots,x_s\geq 1\\ x_1^k+\cdots+x_s^k\leq N-x_{s+1}^k}} e(\alpha(x_1^k+\cdots+x_s^k)) \\ &=\; \sum_{1\leq x_{s+1}\leq N^{1/k}} e(\alpha x_{s+1}^k) \sum_{m\leq N-x_{s+1}^k} r_{k,s}(m)\, e(\alpha m), \end{aligned} \tag{2.4}$$

we apply the inductive hypothesis, and interchange sums in the main term to get

$$\sum_{n\leq N} r_{k,s+1}(n)\, e(n\alpha) \;=\; C_{k,s} \left(\frac{S_k(q,a)}{q}\right)^s \int_0^N \frac{e(\beta v)}{v^{1-s/k}} \left(\sum_{1\leq x_{s+1}\leq (N-v)^{1/k}} e(\alpha x_{s+1}^k) \right) dv \\ + O(N^{s/k} q^{1/2+\epsilon}(1+N|\beta|)^{1/2}).$$

Estimating the inner sum with Theorem 2.1, we find that

$$\sum_{n\leq N} r_{k,s+1}(n)\, e(n\alpha) \;=\; \frac{C_{k,s}}{k} \left(\frac{S_k(q,a)}{q}\right)^{s+1} I_{k,s+1}(\beta,N) + O(N^{s/k} q^{1/2+\epsilon}(1+N|\beta|)^{1/2}), \tag{2.5}$$

where proceeding exactly as we did for $I_{k,2}(\beta,N)$ above, we can show that

(2.6)
$$I_{k,s+1}(\beta,N) \;:=\; \int_0^N v^{s/k-1} \int_0^{N-v} u^{1/k-1}\, e(\beta(u+v))\, du\, dv \;=\; \frac{\Gamma(s/k)\,\Gamma(1/k)}{\Gamma((s+1)/k)} \int_0^N \frac{e(\beta w)}{w^{1-(s+1)/k}}\, dw$$

by means of the beta-function identity $\int_0^1 z^{s/k-1}\,(1-z)^{1/k-1}\,dz = \Gamma(s/k)\,\Gamma(1/k)\big/\Gamma((s+1)/k)$. Inserting (2.6) into (2.5), and noting that $C_{k,s}\cdot\Gamma(s/k)\,\Gamma(1/k)\big/\Gamma((s+1)/k) \;=\; C_{k,s+1}$ completes the induction step, and with it, the proof of Lemma 2.2. $\square$

Recall that we had defined the weight function ν_N by $\nu_N(0) := 0$, and for integers $d \neq 0$ by

$$\nu_N(d) := r_{k,s}(|d|)\,|d|^{1-s/k}\left(1 - \frac{|d|}{N}\right)_+ = \mathbb{1}_{d\in(-N,N)}\, r_{k,s}(|d|)\,|d|^{1-s/k}\left(1 - \frac{|d|}{N}\right). \tag{2.7}$$

In what follows, we write $A \geq O_+(B)$ (respectively, $A \geq -O_+(B)$) to mean that $A \geq \kappa B$ (respectively, $A \geq -\kappa B$) for some constant $\kappa > 0$ depending at most on k, c and ϵ. We now employ Lemma 2.2 to obtain our main workhorse estimates for the major arcs in our upcoming circle method.

Lemma 2.3. *Fix any odd integer $k > 1$, any even integer $s > 0$, and any $\epsilon > 0$. Uniformly in all real β, and in all integers $N \geq 1$, $q \geq 1$, and a, we have*

$$\widehat{\nu}_N(\alpha) = C_{k,s}\left(\frac{S_k(q,a)}{q}\right)^s N\left(\frac{\sin(\pi N\beta)}{\pi N\beta}\right)^2 + O(N^{1-1/k}q^{1/2+\epsilon}(1+N|\beta|)^{1/2}) \tag{2.8}$$

where $\alpha = a/q + \beta$. As a consequence, we have uniformly in all such β, N, q, a,

$$\widehat{\nu}_N(\alpha) \geq -O_+(N^{1-1/k}\,q^{1/2+\epsilon}\,(1+N|\beta|)^{1/2}). \tag{2.9}$$

Proof of Lemma 2.3. We start by noting that

$$\widehat{\nu}_N(\alpha) = \sum_{-N\leq n\leq N}\nu_N(n)\,e(n\alpha) = 2\mathrm{Re}\left(\sum_{n\leq N}\nu_N(\alpha)\,e(n\alpha)\right). \tag{2.10}$$

Now defining $\psi(t) := t^{1-s/k}\,(1 - t/N)$, we see that

$$\sum_{n\leq N}\nu_N(n)\,e(n\alpha) = \sum_{n\leq N} r_{k,s}(n)e(n\alpha)\,\psi(n) = -\int_0^N\left(\sum_{n\leq t} r_{k,s}(n)e(n\alpha)\right)\cdot\psi'(t)\,dt, \tag{2.11}$$

where we have used partial summation and noted that $\psi(N) = 0$. Inserting Lemma 2.2, and noting that $\psi'(t) = (1 - s/k)\,t^{-s/k} - (2 - s/k)\,t^{1-s/k}/N \ll t^{-s/k}$ uniformly in $t \in [1, N]$, we obtain

$$\begin{aligned}\sum_{n\leq N}\nu_N(n)\,e(n\alpha) \\ = C_{k,s}\left(\frac{S_k(q,a)}{q}\right)^s\int_0^N\left(\int_0^t w^{s/k-1}\,e(\beta w)\,dw\right)\cdot\left\{\left(2-\frac{s}{k}\right)\frac{t^{1-s/k}}{N} - \left(1-\frac{s}{k}\right)t^{-s/k}\right\}dt \\ + O\left(q^{1/2+\epsilon}\int_0^N t^{(s-1)/k}\,(1+t|\beta|)^{1/2}\cdot t^{-s/k}dt\right)\end{aligned} \tag{2.12}$$

The total error term in (2.12) is

$$\ll q^{1/2+\epsilon}\,(1+N|\beta|)^{1/2}\int_0^N t^{-1/k}\,dt \ll N^{1-1/k}\,q^{1/2+\epsilon}\,(1+N|\beta|)^{1/2}. \tag{2.13}$$

On the other hand, the main term of (2.12) equals, upon an interchanging of integrals,

$$\begin{aligned} &C_{k,s}\left(\frac{S_k(q,a)}{q}\right)^s \int_0^N w^{s/k-1}\, e(\beta w) \int_w^N \left\{\left(2-\frac{s}{k}\right)\frac{t^{1-s/k}}{N} - \left(1-\frac{s}{k}\right) t^{-s/k}\right\} dt\, dw \\ &= C_{k,s}\left(\frac{S_k(q,a)}{q}\right)^s \int_0^N e(\beta w)\,\left(1-\frac{w}{N}\right) dw. \end{aligned} \tag{2.14}$$

Inserting (2.14) and (2.13) into (2.12), we obtain

$$\sum_{n\le N} \nu_N(n)\, e(n\alpha) \;=\; C_{k,s}\left(\frac{S_k(q,a)}{q}\right)^s \int_0^N e(\beta w)\,\left(1-\frac{w}{N}\right) dw + O\left(N^{1-1/k}\, q^{1/2+\epsilon}\,(1+N|\beta|)^{1/2}\right).$$

The first assertion (2.8) now follows by inserting the above estimate into (2.10), using (1.6), and then noting that the sum $S_k(q,a) = \sum_{r \bmod q} e(ar^k/q)$ is real, as observed in (1.7). The second assertion also follows from the first, since $S_k(q,a)^s \in \mathbb{R}^+$ by the even parity of s. ☐

We conclude this section with the following bound, which will serve as our workhorse estimate for the minor arcs. Given a positive integer b, recall that $\sigma_b > 0$ was defined to be any constant (depending only on b) such that for any fixed $\delta > 0$, a Weyl estimate of the form

$$\sum_{m\le X} e(\alpha m^b) \ll_{b,\delta} X^{1+\delta}\left(\frac{1}{q}+\frac{1}{X}+\frac{q}{X^b}\right)^{\sigma_b} \tag{2.15}$$

holds true for all $X > 1$. As observed in the introduction, we have $\sigma_b \le 1/b$.

Lemma 2.4. *Fix any positive integers b, s, and any $\delta > 0$. Uniformly in $\alpha \in \mathbb{R}$ and $T > 1$, and uniformly in $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying $|\alpha - a/q| \le 1/q^2$, we have*

$$\sum_{n\le T} r_{b,s}(n) e(n\alpha) \;=\; \sum_{\substack{x_1,\dots,x_s\ge 1\\ x_1^b+\cdots+x_s^b\le T}} e(\alpha(x_1^b+\cdots+x_s^b)) \ll_{b,\,s,\,\delta} T^{(s+\delta)/b}\left(\frac{1}{q}+\frac{1}{T^{1/b}}+\frac{q}{T}\right)^{\sigma_b}. \tag{2.16}$$

Consequently for any fixed $\kappa > 0$, we also have

$$\sum_{n\le T} r_{b,s}(n) e(n\alpha)\cdot n^\kappa\left(1-\frac{n}{T}\right) \ll_{b,\,s,\,\delta,\,\kappa} T^{(s+\delta)/b\,+\,\kappa}\left(\frac{1}{q}+\frac{1}{T^{1/b}}+\frac{q}{T}\right)^{\sigma_b}, \tag{2.17}$$

uniformly in all real $T > 1$, and in all $\alpha \in \mathbb{R}$, $q \in \mathbb{N}$, $a \in \mathbb{Z}$ satisfying $|\alpha - a/q| \le 1/q^2$.

Proof. We start by observing that for any $m \in \mathbb{N}$, $\rho \in [0,1]$, and $A_1, \dots, A_m > 0$, we have

$$\frac{A_1^\rho + \ldots + A_m^\rho}{m} \le (A_1 + \ldots + A_m)^\rho \le A_1^\rho + \ldots + A_m^\rho. \tag{2.18}$$

Indeed, the left inequality is immediate from the fact that $(A_1+\ldots+A_m)^\rho \ge \max\{A_1^\rho, \dots, A_m^\rho\}$. To see the inequality on the right, we note that for any $\rho \in (0,1)$ and $t > 0$, the function $h_\rho(t) := (1+t^\rho) - (1+t)^\rho$ has derivative $h_\rho'(t) = \rho t^{\rho-1}\cdot\left(1-(t/(t+1))^{1-\rho}\right) > 0$, so that h_ρ is strictly increasing on the positive real line. As such, $h_\rho(t) > h_\rho(0) = 0$ for all $t > 0$, yielding $(1+t)^\rho \le 1 + t^\rho$ for all such t and for all $\rho \in [0,1]$. Taking $t := A_1/A_2$, we obtain $(A_1+A_2)^\rho \le A_1^\rho + A_2^\rho$ for any $A_1, A_2 > 0$ and any $\rho \in [0,1]$. Inducting on this establishes the second inequality in (2.18).

In the rest of our argument, the implied constants depend at most on b, s and δ for (2.16) (respectively, on b, s, κ and δ for (2.17)). To show the first assertion (2.16), we induct on s, with the base case $s = 1$ just being (2.15). Assuming (2.16) for some fixed s, we start by writing as in (2.4),

$$\left|\sum_{n\le T} r_{b,s+1}(n)e(n\alpha)\right| \le \sum_{1\le x_{s+1}<T^{1/b}} \left|\sum_{\substack{x_1,\dots,x_s\ge 1\\ x_1^k+\cdots+x_s^k\le N-x_{s+1}^k}} e(\alpha(x_1^k+\cdots+x_s^k))\right|,$$

and then invoking the induction step, in conjunction with (2.18) to obtain (with $x \coloneqq x_{s+1}$)

$$\begin{aligned}\left|\sum_{n\le T} r_{b,s+1}(n)e(n\alpha)\right| &\ll \sum_{1\le x<T^{1/b}} (T-x^b)^{(s+\delta)/b}\left(\frac{1}{q}+\frac{1}{(T-x^b)^{1/b}}+\frac{q}{T-x^b}\right)^{\sigma_b}\\ &\le \frac{1}{q^{\sigma_b}}\sum_{1\le x<T^{1/b}}(T-x^b)^{(s+\delta)/b}+\sum_{1\le x<T^{1/b}}(T-x^b)^{(s+\delta-\sigma_b)/b}\\ &\qquad + q^{\sigma_b}\sum_{1\le x<T^{1/b}}(T-x^b)^{(s+\delta)/b-\sigma_b}\end{aligned}$$

Now since $0 < T - x^b \le T$ and $\sigma_b \in (0, 1/b]$, the three sums in the last expression above are at most $T^{(s+1+\delta)/b}$, $T^{(s+1+\delta-\sigma_b)/b}$ and $T^{(s+1+\delta)/b-\sigma_b}$, respectively. This yields

$$\left|\sum_{n\le T} r_{b,s+1}(n)e(n\alpha)\right| \ll T^{(s+1+\delta)/b}\left(\frac{1}{q^{\sigma_b}}+\frac{1}{T^{\sigma_b/b}}+\frac{q^{\sigma_b}}{T^{\sigma_b}}\right) \ll T^{(s+1+\delta)/b}\left(\frac{1}{q}+\frac{1}{T^{1/b}}+\frac{q}{T}\right)^{\sigma_b},$$

where in the last step above, we have finally used (2.18). This completes the induction step, thereby establishing the first assertion (2.16) of the lemma.

In order to obtain (2.17), we apply partial summation (as in (2.11)) to see that

$$\begin{aligned}\sum_{n\le T} r_{b,s}(n)e(n\alpha)\cdot\left(n^\kappa-\frac{n^{1+\kappa}}{T}\right) &= \int_1^T\left(\sum_{n\le t} r_{b,s}(n)e(n\alpha)\right)\cdot\left\{\frac{(1+\kappa)\,t^\kappa}{T}-\kappa\, t^{\kappa-1}\right\}dt\\ &\ll \int_1^T t^{(s+\delta)/b+\kappa-1}\left(\frac{1}{q^{\sigma_b}}+\frac{1}{t^{\sigma_b/b}}+\frac{q^{\sigma_b}}{t^{\sigma_b}}\right)dt,\end{aligned}$$

where in the last line, we have used (2.16), (2.18), and that $t^\kappa/T \le t^{\kappa-1}$ for all $t \in [1, T]$. Distributing the $t^{(s+\delta)/b+\kappa-1}$ and integrating the three terms in the last expression individually, we obtain

$$\sum_{n\le T} r_{b,s}(n)e(n\alpha)\cdot\left(n^\kappa-\frac{n^{1+\kappa}}{T}\right) \ll T^{(s+\delta)/b+\kappa}\left(\frac{1}{q^{\sigma_b}}+\frac{1}{T^{\sigma_b/b}}+\frac{q^{\sigma_b}}{T^{\sigma_b}}\right),$$

which by a final application of (2.18), concludes the proof of the lemma. $\square$

3. Completing the proof of Theorem 1.1

3.1. **The major arcs.** We always assume that N is any positive integer, sufficiently large in terms of k. In the spirit of the classical circle method, we define the major arcs by

$$\mathcal{M}(q,a) \coloneqq \left\{ \alpha \in [0,1] : \left| \alpha - \frac{a}{q} \right| \le \frac{1}{qN^{1-1/k}} \right\}, \ 1 \le q \le N^{1/k}, \ 0 \le a < q, \ \gcd(a,q) = 1.$$

If any real number α lies in both $\mathcal{M}(q,a)$ and $\mathcal{M}(q',a')$ for any q,a,q',a' satisfying $1 \le q, q' \le N^{1/k}$, $0 \le a < q$, $0 \le a' < q'$, and $\gcd(a,q) = \gcd(a',q') = 1$, then by the triangle inequality,

$$\frac{1}{N^{1-1/k}} \left(\frac{1}{q} + \frac{1}{q'} \right) \ge \left| \alpha - \frac{a}{q} \right| + \left| \alpha - \frac{a'}{q'} \right| \ge \left| \frac{a}{q} - \frac{a'}{q'} \right| = \frac{|a'q - aq'|}{qq'} \ge \frac{1}{qq'},$$

leading to $N^{1-1/k} \le q + q' \le 2N^{1/k}$, which is false for all $N \gg_k 1$. This shows that the union

$$\mathcal{M} \coloneqq \bigcup_{1 \le q \le N^{1/k}} \ \bigcup_{\substack{0 \le a < q \\ \gcd(a,q)=1}} \mathcal{M}(q,a)$$

is genuinely a *disjoint* union, so that

$$\int_{\mathcal{M}} |\widehat{\mathbb{1}}_A(\alpha)|^2 \, \widehat{\nu}_N(\alpha) \, d\alpha \ = \sum_{1 \le q \le N^{1/k}} \ \sum_{\substack{0 \le a < q \\ \gcd(a,q)=1}} \int_{a/q \, - \, 1/(q\,N^{1-1/k})}^{a/q \, + \, 1/(q\,N^{1-1/k})} |\widehat{\mathbb{1}}_A(\alpha)|^2 \, \widehat{\nu}_N(\alpha) \, d\alpha. \tag{3.1}$$

Uniformly in α occurring in the inner integral above, we have $|\alpha - a/q| \ \le \ 1/(q\,N^{1-1/k})$ and $1 \le q \le N^{1/k}$. As such, (2.9) yields

$$\begin{aligned} \widehat{\nu}_N(\alpha) \ &\ge \ -O_+\left(N^{1-1/k} \, q^{1/2+\epsilon} \left(1 + \frac{N}{qN^{1-1/k}} \right)^{1/2} \right) \ \ge \ -O_+\left(N^{1-1/k} \, q^{\epsilon} \, \left(q + N^{1/k} \right)^{1/2} \right) \\ &\ge \ -O_+\left(N^{1-1/k} \, (N^{1/k})^{\epsilon} \, \left(N^{1/k} + N^{1/k} \right)^{1/2} \right) \ \ge \ -O_+(N^{1+(-0.5\,+\,\epsilon)/k}). \end{aligned}$$

We use this for all $\alpha \in \mathcal{M} \setminus [0, N^{-1-\epsilon/k}]$, and then note that $\int_{\mathcal{M} \setminus [0,N^{-1-\epsilon/k}]} |\widehat{\mathbb{1}}_A(\alpha)|^2 \, d\alpha$ is at most

$$\int_0^1 |\widehat{\mathbb{1}}_A(\alpha)|^2 \, d\alpha \ = \int_0^1 \sum_{m,n \in A} e((m-n)\alpha) \, d\alpha \ = \sum_{m,n \in A} \int_0^1 e((m-n)\alpha) \, d\alpha \ = \sum_{m \in A} 1 \ = \ |A|, \tag{3.2}$$

Doing this, we obtain from (3.1),

$$\int_{\mathcal{M}} |\widehat{\mathbb{1}}_A(\alpha)|^2 \, \widehat{\nu}_N(\alpha) \, d\alpha \ \ge \int_0^{N^{-1-\epsilon/k}} |\widehat{\mathbb{1}}_A(\alpha)|^2 \, \widehat{\nu}_N(\alpha) \, d\alpha \ - \ O_+(N^{1+(-0.5\,+\,\epsilon)/k}). \tag{3.3}$$

To estimate the first integral on the right above, note that uniformly in all $\alpha \in (0, N^{-1-\epsilon/k}]$, we have

$$\begin{aligned} |\widehat{\mathbb{1}}_A(\alpha)|^2 &= \sum_{m,n \in A} e((m-n)\alpha) \ = \sum_{m,n \in A} \{1 + O(|m-n|\alpha)\} \\ &= \ |A|^2 + O(|A|^2 N\alpha) \ \ge \ |A|^2 \ - \ O_+(|A|^2 \, N^{-\epsilon/k}) \ \ge \ |A|^2/2. \end{aligned} \tag{3.4}$$

Moreover, applying (2.8) to $q = 1$ and $a = 0$, we also have uniformly in all $\alpha \in (0, N^{-1-\epsilon/k}] \subseteq \mathcal{M}(0,1)$,

$$\widehat{\nu}_N(\alpha) = C_{k,s}\, N\, \left(\frac{\sin(\pi N\alpha)}{\pi N\alpha}\right)^2 + O\left(N^{1-1/k}\left(1 + \frac{N}{N^{1+\epsilon/k}}\right)^{1/2}\right)$$

Since $\sin(\pi N\alpha) = \pi N\alpha\big(1 + O(N^2\alpha^2)\big) = \pi N\alpha\big(1 + O(N^{-2\epsilon/k})\big) \geq \pi N\alpha/2$ uniformly in $\alpha \in (0, N^{-1-\epsilon/k}]$, it follows from the above display that

$$\widehat{\nu}_N(\alpha) \gg N. \tag{3.5}$$

Inserting (3.4) and (3.5) into (3.3), we obtain the following clean lower bound for the total contribution of all our major arcs

$$\int_{\mathcal{M}} |\widehat{\mathbb{1}}_A(\alpha)|^2\, \widehat{\nu}_N(\alpha)\, d\alpha \gg |A|^2\, N^{-\epsilon/k} - O_+(N^{1+(-0.5+\epsilon)/k}\, |A|). \tag{3.6}$$

3.2. **The minor arcs.** For any $\alpha \in [0,1]$, Dirichlet's approximation theorem shows that there exist integers a, q with $1 \leq q \leq N^{1-1/k}$, $0 \leq a < q$ and $\gcd(a,q) = 1$, such that $|\alpha - a/q| \leq 1/(qN^{1-1/k})$. If $\alpha \in [0,1] \setminus \mathcal{M}$, then we must also have $q > N^{1/k}$. Hence $q \in (N^{1/k}, N^{1-1/k}]$, so that from (2.17), we obtain

$$\widehat{\nu}_N(\alpha) = 2\mathrm{Re} \sum_{n \leq N} r_{k,s}(n) e(n\alpha)\, n^{1-s/k}\left(1 - \frac{n}{N}\right) \ll N^{1+\epsilon/k}\left(\frac{1}{q} + \frac{1}{N^{1/k}} + \frac{q}{N}\right)^{\sigma_k} \ll N^{1+(-\sigma_k+\epsilon)/k},$$

uniformly in $\alpha \in [0,1] \setminus \mathcal{M}$, with σ_k as defined before Theorem 1.1. Consequently, by (3.2),

$$\int_{[0,1]\setminus\mathcal{M}} |\widehat{\mathbb{1}}_A(\alpha)|^2\, \widehat{\nu}_N(\alpha)\, d\alpha \geq -O_+(N^{1+(-\sigma_k+\epsilon)/k}\, |A|). \tag{3.7}$$

Finally, if $(A - A) \cap \mathcal{A}_{k,s} = \emptyset$, then $\int_0^1 |\widehat{\mathbb{1}}_A(\alpha)|^2\, \widehat{\nu}_N(\alpha)\, d\alpha = 0$ by (1.4). As such, from (3.6) and (3.7), we obtain

$$|A| \ll N^{1+(-0.5+2\epsilon)/k} + N^{1+(-\sigma_k+2\epsilon)/k} \ll N^{1-\tau_k+\epsilon},$$

where indeed $\tau_k = \sigma_k/k \leq 1/(2k)$. This concludes the proof of Theorem 1.1. □

Remark. It is worth noting that the entire argument goes through if we replace $\nu_N(d)$ by the more general weight function

$$\mu_{k,N}(d) := \mu_{k,N}^{(c)}(d) := r_{k,s}(|d|)\, |d|^c\, \left(1 - \frac{|d|}{N}\right)_+$$

for any constant $c \in (-s/k, 1 - s/k]$. Indeed the analogues of (2.8) and (2.9) are

(3.8)
$$\widehat{\mu}_N(\alpha) = C_{k,s}\left(\frac{S_k(q,a)}{q}\right)^s \int_0^N \frac{\cos(2\pi\beta w)}{w^{1-s/k-c}}\left(1 - \frac{w}{N}\right) dw + O(N^{(s-1)/k+c}\, q^{1/2+\epsilon}\, (1 + N|\beta|)^{1/2})$$

(3.9)
$$\geq -O_+(N^{(s-1)/k+c}\, q^{1/2+\epsilon}\, (1 + N|\beta|)^{1/2}).$$

Here, we cannot evaluate the main term in (3.8) directly. Instead, to deduce (3.9), we show the non–negativity of the integral in the main term of (3.8) by integrating by parts twice, first time with the antiderivative $\sin(2\pi\beta w)/2\pi\beta$ of the function $\cos(2\pi\beta w)$, and the second time with the antiderivative $(1-\cos(2\pi\beta w))/2\pi\beta$ of the function $\sin(2\pi\beta w)$.

Another consequence of (3.8) is that uniformly in $w \in [0, N]$ and in $\alpha \in (0, N^{-1-\epsilon/k}]$, the estimate $\cos(2\pi\alpha w) = 1 + O(\alpha^2 w^2) = 1 + O(N^{-2\epsilon/k}) \geq 1/2$ gives $\widehat{\mu}_N(\alpha) \gg N^{s/k+c}$, which is the desired generalization of (3.5). The rest of the argument is entirely analogous, and the main term and error terms all get scaled suitably by the same factor, hence we still end up with the same bound on $|A|$.

Acknowledgements

This project arose out of discussions at the Simons Summer School and Workshop on Discrete Harmonic Analysis and Ramsey Theory, which took place at the Renyi Institute from May 11th to June 22nd. The authors would like to thank the organizers for their meticulous hardwork in organizing these events. The authors would also like to thank Máté Matolcsi and Imre Ruzsa for useful discussions and for their interest in the problem. The third author was partially supported from the Simons Foundation grant Grant MPS-TSM-854813 and NFKIH Excellence 15421. The fourth author acknowledges support from the Charles University Grant PRIMUS/25/SCI/008.

Statement on AI use

The regular version of GPT 5.5 was used to simplify the original argument given for (2.16). Besides that, there has been no use of artificial intelligence in the production of this manuscript and the authors take full responsibility for the content.

Conflict of interest

The authors declare no conflict of interest.

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