CYCLIC-QUADRILATERALS AND GOWERS' U^3-NORM

Abstract. Let $\Gamma_0 = \{v_0, v_1, v_2, v_3\} \subseteq \mathbb{R}^d$ be a cyclic quadrilateral spanning a two-dimensional subspace. In this note we prove a generalized von Neumann inequality which controls the normalized multilinear count of isometric copies of $\lambda\Gamma_0$ by a standard local Gowers U^3 norm at a scale $\varepsilon\lambda$, up to a small error. The argument provides a model for applying higher-order Fourier analysis to spherical configuration problems in Euclidean Ramsey theory.

1. Introduction

Throughout the paper, $Q_d = [0,1]^d$. We call a 4-point configuration $\Gamma_0 = \{v_0 = 0, v_1, v_2, v_3\} \subseteq \mathbb{R}^d$ a circular quadrilateral if it spans a 2-dimensional subspace and is contained in sphere. Then v_1, v_2 are not collinear and $v_3 = a_1v_1 + a_2v_2$ for some $a_1, a_2 \in \mathbb{R}$. Write $t_1 = |v_1|^2,\ t_2 = |v_2|^2$ and $t_{12} = |v_1 - v_2|^2$. Let $\Gamma = \{x, x + x_1, x + x_2, x + x_3\} \subseteq \mathbb{R}^d$ be a quadrilateral. We say the Γ is isometric to Γ_0 if and $x_3 = a_1x_1 + a_2x_2$ and $x = (x_1, x_2) \in S_{\mathcal{F}}$, the zero set of the family of functions

$$\mathcal{F} = \{|x_1|^2 - t_1,\ |x_2|^2 - t_2,\ |x_1 - x_2|^2 - t_{12}\}.$$

This implies that the Γ can be obtained from Γ_0 via a rotation and translation.

Let $A \subseteq Q_d$ be a (Borel) measurable set and let $g = \mathbf{1}_A$ denote its indicator function. For given $\lambda > 0$, define

$$\mathcal{N}_\lambda(A) := \int_{\mathbb{R}^d} \int_{S_{\mathcal{F}}} g(x)g(x + \lambda x_1)g(x + \lambda x_2)g(x + a_1x_1 + a_2x_2)\, d\omega_{\mathcal{F}}(x_1, x_2)\, dx$$

where $d\omega_{\mathcal{F}}$ is the so-called Leray measure on the on the configuration variety $S_{\mathcal{F}}$. This is a positive measure on the non-singular surface $S_{\mathcal{F}}$ defined in (4.3). The above expression is a normalized count of the isometric embeddings of $\lambda\Gamma_0$ into the set A.

More generally given measurable functions $g_0, \dots, g_3 : [0,1]^d \to [-1,1]$ define the multilinear expression

$$\mathcal{N}_\lambda(g_0, g_1, g_2, g_3) := \int_{\mathbb{R}^d} \int_{S_{\mathcal{F}}} g_0(x)g_1(x + \lambda x_1)g_2(x + \lambda x_2)g_3(x + \lambda(a_1x_1 + a_2x_2))\, d\omega_{\mathcal{F}}(x_1, x_2)\, dx.$$

We recall the local Gowers norm $\|f\|_{U^3(L)}$ of a compactly supported function bounded function $f : \mathbb{R}^n \to \mathbb{R}$

$$\|f\|_{U^3(L)}^8 := \int \Delta_a\Delta_b\Delta_c f(x)\, \vartheta_L(a)\vartheta_L(b)\vartheta_L(c)\, dx\, da\, db\, dc,$$

where

$$\vartheta = \zeta * \widetilde{\zeta}, \qquad \widetilde{\zeta}(y) = \zeta(-y), \qquad \vartheta_L(y) = L^{-d}\vartheta(y/L),$$

for some smooth bump function ζ. The assumption on ϑ guarantees the positivity of the above expression and the fact that it defines a norm, see Section 3.

The main result of this note is the following.

Theorem 1.1 (Main theorem)**.** *Let $d \geq 14$ and let $\Gamma_0 \subseteq \mathbb{R}^d$ be a fixed cyclic quadrilateral spanning a two-dimensional subspace. There are constants $c, C > 0$, depending only on d and Γ_0, such that for every $0 < \lambda \leq 1$, every $0 < \varepsilon < 1$, and all measurable functions $g_i : Q_d \to [-1, 1]$, $0 \leq i \leq 3$, one has*

$$|\mathcal{N}_\lambda(g_0, g_1, g_2, g_3)| \leq C\varepsilon^{-1/68} \min_{0 \leq i \leq 3} \|g_i\|_{U^3(\varepsilon\lambda)}^{1/32} + C\varepsilon^{1/68}, \tag{1.1}$$

for some constant $C > 0$ that may depend on dimension n and the fixed quadrilateral Γ_0.

The numerical exponents in (3) are not optimized; the important point is that the local scale and all losses depend on the error parameter through fixed powers.

Our approach proceeds in three stages. First we majorize the multilinear expression which counts isometric embeddings of the quadrilateral Γ_0 in A by an expression which may be viewed as a weighted local geometric analogue of the Gowers U^2 norm. We then remove the configuration-dependent weight by a smoothing argument at a smaller scale. The final step is to majorize the resulting geometric $U^2_\perp$ norm by a standard local Gowers U^3 norm.

1.1. **Counting distances.** Let us we illustrate our method on the simple example of counting distances. Let $d \geq 4$, let $0 < \lambda \leq 1$, and write formally

$$d\sigma_\lambda(y) := \lambda^{-(d-2)} \delta\big(|y|^2 - \lambda^2\big)\, dy$$

for the normalized surface area measure on the sphere of radius λ. For real-valued functions $f_0, f_1 : \mathbb{R}^d \to [-1, 1]$, supported in a fixed bounded set, let

$$\mathcal{N}_\lambda(f_0, f_1) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f_0(x) f_1(x + y)\, d\sigma_\lambda(y)\, dx.$$

Using the notation

$$\Delta_h f(x) := f(x) f(x + h),$$

an application of the Cauchy–Schwarz inequality in the x-variable gives

$$|\mathcal{N}_\lambda(f_0, f_1)|^2 \lesssim \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta_h f_1(x) W_\lambda(h)\, dh\, dx, \tag{1.2}$$

where formally

$$W_\lambda(h) := \lambda^{-2(d-2)} \int_{\mathbb{R}^d} \delta\big(|y|^2 - \lambda^2\big) \delta\big(|y + h|^2 - \lambda^2\big)\, dy. \tag{1.3}$$

More precisely $W_\lambda(h)$ is the Leray measure of the intersection

$$S_{\lambda,h} := \left\{ y : \ |y|^2 = \lambda^2, \quad y \cdot h = -\frac{|h|^2}{2} \right\}.$$

For $0 < |h| < 2\lambda$, this is a $(d-2)$-dimensional sphere of radius $r = \big(\lambda^2 - \frac{|h|^2}{4}\big)^{1/2}$, and direct calculation of its Leray measure described in Section 4, gives

$$W_\lambda(h) = c_d \lambda^{-d} \frac{\left(1 - \frac{|h|^2}{4\lambda^2}\right)_+^{(d-3)/2}}{|h|/\lambda}. \tag{1.4}$$

In particular,

$$\|W_\lambda\|_{L^1(\mathbb{R}^d)} \lesssim 1, \qquad |\nabla W_\lambda|_{L^1(\mathbb{R}^d)} \lesssim \lambda^{-1}. \tag{1.5}$$

The right side of (1.2) is a weighted local U^1 quantity. We now compare it with a standard local U^1-norm at a smaller scale. Fix a nonnegative function $\zeta \in C_c^\infty(\mathbb{R}^d)$ with $\int \zeta = 1$, let

$$\widetilde{\zeta}(u) := \zeta(-u), \qquad \vartheta := \zeta * \widetilde{\zeta}, \qquad \vartheta_L(t) := L^{-d} \vartheta(t/L),$$

and define

$$\|f\|_{U^1(L)}^2 := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta_t f(x) \vartheta_L(t)\, dt\, dx. \tag{1.6}$$

This quantity is nonnegative. Indeed, if

$$F_L(x) := \int_{\mathbb{R}^d} f(x-u)\zeta_L(u)\, du, \qquad \zeta_L(u) := L^{-d}\zeta(u/L),$$

then it is easy to see that

$$\|f\|_{U^1(L)}^2 = \int_{\mathbb{R}^d} |F_L(x)|^2\, dx. \tag{1.7}$$

Proposition 1.1 (Weighted versus standard local U^1)**.** *Let $0 < L \le \lambda \le 1$, and let $f : \mathbb{R}^d \to [-1,1]$ be supported in a fixed bounded set. Then*

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta_h f(x) W_\lambda(h)\, dh\, dx \right| \lesssim \|f\|_{U^1(L)}^2 + \frac{L}{\lambda}. \tag{1.8}$$

Proof. Let

$$W_{\lambda,L} := W_\lambda * \vartheta_L.$$

From (1.5) we have

$$\begin{aligned} \|W_{\lambda,L} - W_\lambda\|_1 &\le \int \vartheta_L(t) \|W_\lambda(\,\cdot - t) - W_\lambda\|_1\, dt \\ &\lesssim \|\nabla W_\lambda\|_1 \int |t| \vartheta_L(t)\, dt \lesssim \frac{L}{\lambda}. \end{aligned} \tag{1.9}$$

Since $|f| \le 1$ and f has bounded support, replacing W_λ by $W_{\lambda,L}$ changes the left side of (1.2) by $O(L/\lambda)$.

Let

$$C_f(h) := \int_{\mathbb{R}^d} \Delta_h f(x)\, dx.$$

Using $\vartheta_L = \zeta_L * \widetilde{\zeta}_L$ and Fubini's theorem, we obtain

$$\int C_f(h) W_{\lambda,L}(h)\, dh = \int W_\lambda(H) \left(\int \vartheta_L(t) C_f(H+t)\, dt \right) dH \tag{1.10}$$

$$= \int W_\lambda(H) \left(\int F_L(x) F_L(x+H)\, dx \right) dH. \tag{1.11}$$

By Cauchy–Schwarz and (1.7),

$$\left| \int F_L(x) F_L(x+H)\, dx \right| \le \|F_L\|_2^2 = \|f\|_{U^1(L)}^2$$

uniformly in H. The estimate now follows from $\|W_\lambda\|_1 \lesssim 1$. □

Combining (1.2) with Proposition 1.1, and using the symmetry between the two functions, gives

$$|\mathcal{N}_\lambda(f_0, f_1)|^2 \lesssim \min_{i=0,1} \|f_i\|_{U^1(L)}^2 + \frac{L}{\lambda}. \tag{1.12}$$

Consequently,

$$|\mathcal{N}_\lambda(f_0, f_1)| \lesssim \min_{i=0,1} \|f_i\|_{U^1(L)} + \left(\frac{L}{\lambda}\right)^{1/2}. \tag{1.13}$$

In particular, taking $L=\varepsilon^2\lambda$ yields

$$|\mathcal{N}_\lambda(f_0,f_1)| \lesssim \min_{i=0,1} \|f_i\|_{U^1(\varepsilon^2\lambda)} + O(\varepsilon). \tag{1.14}$$

2. Counting isometric copies of quadrilaterals.

The above elementary argument already contains some of the basic ideas used for circular quadrilaterals. Applying the Cauchy–Schwarz inequalities twice produces a geometric uniformity quantity with a singular weight function at scale λ. Averaging the weight over translations of size $L \ll \lambda$ replaces it by a standard local geometric U^2-type uniformity norm at scale L, with an error controlled by the variation of the weight. For quadrilaterals the increment variables h,k lie on the hypersurface $h\cdot k=0$, so the analogous smoothing step also requires control of the associated Leray density and of the singular two-variable weight $W(h,k)$ simultaneously.

2.1. **The configuration space of $\lambda\Gamma_0$.** We call a 4-point configuration $\Gamma_0=\{v_0=0,v_1,v_2,v_3\}\subseteq \mathbb{R}^d$ a circular quadrilateral if it spans a 2-dimensional subspace and is contained in sphere. This implies that there exists nonzero real numbers a_0,a_1,a_2 so that

$$a_0v_0+a_1v_1+a_2v_2=v_3 \quad \textit{and} \quad a_0+a_1+a_2=1. \tag{2.1}$$

Moreover, there is a point u so that $|v_i-u|^2=r^2$ for $i=0,\dots,3$, thus

$$a_0|v_0-u|^2+a_1|v_1-u|^2+a_2|v_2-u|^2=|v_3-u|^2, \tag{2.2}$$

thus it follows from (2.1)

$$a_0|v_0|^2+a_1|v_1|^2+a_2|v_2|^2=|v_3|^2. \tag{2.3}$$

It is simple but important observation that (2.1) and (2.3) implies (2.2) for *any* $u\in\mathbb{R}^d$. In particular if (2.1) and (2.3) holds then Γ_0 must be spherical as there is a point u for which $|v_i-u|^2=r^2$ for $i=0,1,2$ and hence $|v_3-u|^2=r^2$ as well. Note that, since $v_0=0$, equations (2.1) and (2.3) take the form

$$a_1v_1+a_2v_2=v_3 \quad \textit{and} \quad a_1|v_1|^2+a_2|v_2|^2=|v_3|^2.$$

Assume $d>2$ and let $\lambda>0$. A 4-point configuration $\Gamma=\{x_0,x_0+x_1,x_0+x_2,x_0+x_3\}$ is isometric to $\lambda\Gamma_0$, written $\Gamma\simeq\lambda\Gamma_0$, if $|x_1|^2=\lambda^2|v_1|^2$, $|x_2|^2=\lambda^2|v_2|^2$, $|x_1-x_2|^2=\lambda^2|v_1-v_2|^2$ and $x_3=a_1x_1+a_2x_2$.
Write $t_i:=|v_i|^2$ for $i=1,2$ and $t_{12}:=|v_1-v_2|^2$.

Let $\Gamma=\{0,x_1,x_2,x_3\}$, then $\Gamma\simeq\lambda\Gamma_0$ if and only if $x=(x_1,x_2,x_3)\in S_{\mathcal{F}_\lambda}$, the zero set of the family of functions

$$\mathcal{F}'_\lambda=\{l(x),\, q_1(x),\, q_2(x),\, q_{12}(x)\}, \tag{2.4}$$

where $l(\underline{x})=a_1x_1+a_2x_2-x_3$, $q_1(\underline{x})=|x_1|^2-\lambda^2\,t_1$, $q_2(\underline{x})=|x_2|^2-\lambda^2\,t_2$, $q_{12}(\underline{x})=|x_1-x_2|^2-\lambda^2t_{12}$.

As explained in Sec. 7.2, a point $x=(x_1,x_2,x_3)\in S_{\mathcal{F}'_\lambda}$ is non-singular if and only if $x=(x_1,x_2)\in S_{\mathcal{F}_\lambda}$ is non-singular where

$$\mathcal{F}_\lambda=\{q_1(x),\ q_2(x),\ q_{12}(x)\}. \tag{2.5}$$

Using the language of Section 7.1, we have

Lemma 2.1. *The partition of coordinates $z=x_1$, $y=x_2$ is* admissible *for the system $\mathcal{F}_\lambda$. Moreover the surface $S_{\mathcal{F}_\lambda}$ is non-singular.*

Proof. If $(x_1,x_2)\in S_{\mathcal{F}_\lambda}$ then $\nabla_{x_1}q_1(x_1)=2x_1\neq 0$, moreover the vectors x_1 and x_2 are linearly independent as the triangle $\{0,x_1,x_2\}$ is isometric to the triangle $\lambda\{0,v_1,v_2\}$. This implies that the gradient vectors $\nabla_{x_2}\,q_2(x_2)=2x_2$ and $\nabla_{x_2}\,q_{12}(x_2)=2(x_2-x_1)$ are also linearly independent. This shows that the partition $y=x_2$, $z=x_1$ is admissible, moreover that $S_{\mathcal{F}'}$ has no singular points. □

Let $A\subseteq[0,1]^d$ be a (Borel) measurable set and let $g=\mathbf{1}_A$ denote its indicator function. Recall

$$\mathcal{N}_\lambda(A):=\lambda^{-2d+6}\int\int_{S_{F'_\lambda}} g(x)g(x+x_1)g(x+x_2)g(x+x_3)\,d\omega_{\mathcal{F}'_\lambda}(x_1,x_2,x_3)\,dx$$

where the system of forms $\mathcal{F}_\lambda$ is given in (2.4) and the measure $d\omega_{\mathcal{F}'_\lambda}$ is defined in (4.3). By (4.15) we have that

$$\mathcal{N}_\lambda(A):=\lambda^{-2d+6}\int\int_{S_{F'_\lambda}} g(x)g(x+x_1)g(x+x_2)g(x+a_1x_1+a_2x_2)\,d\omega_{\mathcal{F}_\lambda}(x_1,x_2)\,dx. \tag{2.6}$$

It is clear that if $\mathcal{N}_\lambda(A)>0$ then A must contain an isometric copy of $\lambda\Gamma_0$. Let $\mathcal{F}_1:=\{p_1(x),p_2(x),p_{12}(x)\}$, with $p_i(x)=|x_i|^2$ and $p_{12}(x)=|x_1-x_2|^2$, $x=(x_1,x_2)$. Then, with notations of Section 7.4 we have that $S_{\mathcal{F}_\lambda}=\lambda S_{\mathcal{F}_1,t}$ with $t=(t_1,t_2,t_{12})$. Since the total degree of the system $\mathcal{F}_0$ is 6, we have by (4.17)

$$\mathcal{N}_\lambda(A):=\int\int_{S_{F'}} g(x)g(x+\lambda x_1)g(x+\lambda x_2)g(x+\lambda(a_1x_1+a_2x_2))\,d\omega_{\mathcal{F}_1}(x_1,x_2)\,dx. \tag{2.7}$$

Note that the expression in (2.7) is normalized i.e. $|\mathcal{N}_\lambda(A)|\lesssim 1$. Scaling with λ shows that (2.6) is the special case of the more general expression,

$$\mathcal{N}_\lambda(g_0,g_1,g_2,g_3):=\int\int_{S_{\mathcal{F}'}} g_0(x)g_1(x+\lambda x_1)g_2(x+\lambda x_2)g_3(x+\lambda(a_1x_1+a_2x_2))\,d\omega_{\mathcal{F}_1}(x_1,x_2)\,dx. \tag{2.8}$$

2.2. **Upper bounds.** We will provide upper bounds for the expression in (2.8) from above which are independent of the geometry of Γ_0. We have $\mathcal{F}_1=\mathcal{F}_1^1\cup\mathcal{F}_1^2$ with $\mathcal{F}_1^1=\{p_1(x_1)=|x_1|^2-t_1\}$ and $\mathcal{F}_1^2=\{p_2(x_2)=|x_2|^2-t_2,\ p_{12}(x_1,x_2)=|x_1-x_2|^2-t_{12}\}$. For simplicity of notation we will write $\mathcal{N}_\lambda:=\mathcal{N}_\lambda(g_0,g_1,g_2,g_3)$ and use the convention $x_3:=a_1x_1+a_2x_2$.

By Lemma 2.1 one may apply (4.7) to obtain

$$|\mathcal{N}_\lambda|\le\int\int|g_0(x)g_1(x+\lambda x_1)|\left|\int g_2(x+\lambda x_2)g_3(x+\lambda x_3)\,d\omega_{\mathcal{F}_{2,x_1}}(x_2)\right|\,d\omega_{\mathcal{F}_1}(x_1)\,dx. \tag{2.9}$$

Since $|g_i|\le 1$ and is supported on $[0,1]^d$, as explained in Sec. 7.2, see (4.8)-(4.12), an application of the Cauchy-Schwarz inequality gives

$$|\mathcal{N}_\lambda|^2\lesssim\int\int g_2(x+\lambda x_2)g_2(x+\lambda y_2)g_3(x+\lambda x_3)g_3(x+\lambda y_3)\,d\omega_{\mathcal{F}^2_{1,x_1}}(x_2)\,d\omega_{\mathcal{F}^2_{1,x_1}}(y_2)\,d\omega_{\mathcal{F}^1_1}(x_1)\,dx, \tag{2.10}$$

with the convention $x_3=a_1x_1+a_2x_2$, $y_3=a_1x_1+a_2y_2$. The above iterated integral may be rewritten as

$$\int\int_{S_{\mathcal{F}^2}} g_2(x+\lambda x_2)g_2(x+\lambda y_2)g_3(x+\lambda x_3)g_3(x+\lambda y_3)\ d\omega_{\mathcal{F}^2}(x_1,x_2,y_2)\,dx \tag{2.11}$$

where $S_{\mathcal{F}^2}\subseteq\mathbb{R}^{3d}$ is the algebraic set defined by the system

$$\mathcal{F}^2=\{p_1(x_1),p_2(x_2),p_2(y_2),p_{12}(x_1,x_2),p_{12}(x_1,y_2)\}.$$

We proceed as in Sec. 7.1, change variables $y_2 := x_2 + h$ which implies $y_3 := x_3 + a_2 h$, and rewrite the system $\mathcal{F}^2$ in the new variables (h, x_1, x_2). We obtain

$$|\mathcal{N}_\lambda|^2 \lesssim \int\int \Delta_{\lambda h}\, g_2(x+\lambda x_2)\, \Delta_{\lambda a_2 h}\, g_3(x+\lambda x_3)\, d\omega_{\mathcal{F}^2}(x_1,x_2,h)\, dx, \tag{2.12}$$

where $\mathcal{F}^2(x_1,x_2,h) = \{p_1(x_1), p_2(x_2), p_2(x_2+h), p_{12}(x_1,x_2), p_{12}(x_1,x_2+h)\}$.
It will be convenient to consider the equivalent system

$$\begin{aligned}\mathcal{F}^3(x_1,x_2,h) := \{&p_1(x_1), p_2(x_2), p_{12}(x_1,x_2),\\ &p_2'(x_2,h) := p_2(x_2+h) - p_2(x_2),\\ &p_{12}'(x_1,h) := \big(p_2(x_2+h)-p_2(x_2)\big) - \big(p_{12}(x_1,x_2+h) - p_{12}(x_1,x_2)\big)\}.\end{aligned}$$

Note that

$$p_2'(x_2,h) = p_2(x_2+h) - p_2(x_2) = 2x_2\cdot h + |h|^2$$

and $p_{12}'(x_1,h) = 2x_1\cdot h$. The passage from $\mathcal{F}^2$ to $\mathcal{F}^3$ consists of row operations (up to reordering the equations) and hence preserves the associated Leray measure.

Relabeling the three nonzero vertices of Γ_0, together with the corresponding functions, parameters t_i, and polynomials p_i, p_{ij}, if necessary, we may assume that $a_2 \neq 1$. Indeed, if $a_2 = 1$ but $a_1 \neq 1$, interchange v_1 and v_2; if $a_1 = a_2 = 1$, use the ordering $\widetilde{v}_1 = v_3$, $\widetilde{v}_2 = v_2$, $\widetilde{v}_3 = v_1$, for which $\widetilde{v}_3 = \widetilde{v}_1 - \widetilde{v}_2$.

Also, this can be done while placing any prescribed function g_i in the g_2-slot. Let $c_0, \ldots, c_3$ be the nonzero coefficients of the affine dependence

$$\sum_{i=0}^{3} c_i v_i = 0, \qquad \sum_{i=0}^{3} c_i = 0.$$

Given a prescribed index j, choose $\ell \neq j$ so that $c_j + c_\ell \neq 0$; such an ℓ exists, since otherwise the preceding two identities would force $c_j = 0$. Using one of the two remaining vertices as the new base point and translating it to the origin, using the other as the new v_1, placing the prescribed vertex in the new v_2-slot, and using v_ℓ as the dependent vertex v_3. Then $a_2 = -c_j/c_\ell$, and hence

$$b = 1 - a_2 = \frac{c_\ell + c_j}{c_\ell} \neq 0.$$

To eliminate the factor $\Delta_{\lambda a_2 h}\, g_3(x+\lambda x_3)$ we observe that $x + \lambda x_3 = x + \lambda a_1 x_1 + \lambda a_2 x_2$ and make a change of variables $x := x + \lambda a_2 x_2$, which does not affect the integral in (2.12). The g_3-factor is then independent of x_2 and may be discarded using $|g_3| \leq 1$. This gives, with $b = 1 - a_2 \neq 0$,

$$|\mathcal{N}_\lambda|^2 \lesssim \int \left| \int \Delta_{\lambda h}\, g_2\,(x+\lambda b x_2)\, d\omega_{\mathcal{F}^3_{x_1,h}}(x_2)\right| d\omega_{\mathcal{F}^3(x_1,h)}\, dx, \tag{2.13}$$

where $\mathcal{F}^3(x_1,h) = \{p_1(x_1), p_{12}'(x_1,h)\}$, and $\mathcal{F}^3_{x_1,h} = \{p_2(x_2), p_2'(x_2,h), p_{12}(x_1,x_2)\}$ for given x_1, h. To justify (2.13) one needs to verify the following.

Lemma 2.2. *Let $d \geq 4$. Then the partition of coordinates $y = x_2$, $z = (x_1, h)$ is admissible for the system $\mathcal{F}^3(h, x_1, x_2)$.*

Proof. Since $S_{\mathcal{F}^3} = S_{\mathcal{F}^2}$, we have that $(x_1,x_2,h) \in S_{\mathcal{F}^3}$ if and only if $\{0, x_1, x_2\} \simeq \{0, v_1, v_2\}$ and also $\{0, x_1, x_2+h\} \simeq \{0, v_1, v_2\}$. This implies that the vectors $\nabla_{(x_1,h)} p_1(x_1,h) = (2x_1, 0)$ and $\nabla_{(x_1,h)} p_{12}'(x_1,h) = (2h, 2x_1)$ are linearly independent at any point $(x_1,x_2,h) \in S_{\mathcal{F}^3}$.

We show that there is a point $(x_1,x_2,h) \in S_{\mathcal{F}^3}$ such that the vectors $\nabla_{x_2} p_2(x_2) = 2x_2$, $\nabla_{x_2} p_{12}(x_1,x_2) = 2(x_2 - x_1)$ and $\nabla_{x_2} p_2'(x_2,h) = 2h$, or equivalently the vectors x_1, x_2, $y_2 = x_2 + h$, are linearly independent. Let $x_1 = v_1$ and $x_2 = v_2$. Then $(x_1,x_2,h) \in S_{\mathcal{F}^3}$ if and only if $|y_2|^2 = t_2$ and

$|y_2 - v_1|^2 = t_{12}$, i.e. if y_2 is contained in the intersection of two spheres. Clearly the intersection is non-empty as it contains v_2; moreover, v_2 is a non-singular point as v_2 is not in the span of the centers 0 and v_1, see Sec. 7.2. This further implies that the intersection is a 2-codimensional sphere which cannot be contained in the 2-dimensional subspace spanned by the vectors v_1 and v_2 in dimensions $d \geq 4$. Thus there exists a point $(x_1, x_2, h) \in S_{\mathcal{F}^3}$ consisting of linearly independent vectors. ☐

Then one can apply the Cauchy-Schwarz inequality again to obtain

$$|\mathcal{N}_\lambda|^4 \lesssim \int \int\int \Delta_{\lambda h}\, g_2\,(x+\lambda b x_2)\, \Delta_{\lambda h}\, g_2\,(x+\lambda b y_2)\, d\omega_{\mathcal{F}^3_{x_1,h}}(x_2)\, d\omega_{\mathcal{F}^3_{x_1,h}}(y_2)\, d\omega_{\mathcal{F}^3(x_1,h)}\, dx. \tag{2.14}$$

Let $y_2 := x_2 + k$ and rewriting the system in the variables h, k, x_1, x_2 we obtain

$$|\mathcal{N}_\lambda|^4 \lesssim \int \int \Delta_{\lambda bk, \lambda h}\, g_2\,(x+\lambda b x_2)\ d\omega_{\mathcal{F}^4(x_1,x_2,h,k)}\, dx, \tag{2.15}$$

where $\Delta_{k,h} g := \Delta_k \Delta_h g$, and, before row reduction, the system consists of the eight equations

$$\{p_1(x_1), p'_{12}(x_1,h), p_2(x_2), p'_2(x_2,h), p_{12}(x_1,x_2),$$
$$p_2(x_2+k), p'_2(x_2+k,h), p_{12}(x_1,x_2+k)\}.$$

Define

$$q(h,k) := p'_2(x_2+k,h) - p'_2(x_2,h) = 2h\cdot k$$

and

$$\begin{aligned} p'_{12}(x_1,k) := &\big(p_2(x_2+k) - p_2(x_2)\big) \\ &- \big(p_{12}(x_1,x_2+k) - p_{12}(x_1,x_2)\big) = 2x_1\cdot k. \end{aligned}$$

After some row operations of the Jacobian of the system we may use the equivalent system

$$\begin{aligned} \mathcal{F}^4 = \{&p_1(x_1), p'_{12}(x_1,h), p'_{12}(x_1,k), q(h,k), \\ &p_2(x_2), p_2(x_2+k), p_{12}(x_1,x_2), p'_2(x_2,h)\}. \end{aligned}$$

By a shift $x := x + \lambda b x_2$, which is allowed as the measure $\omega_{\mathcal{F}^4}$ is independent of x, we have that

$$|\mathcal{N}_\lambda|^4 \lesssim \int \int \Delta_{\lambda bk, \lambda h}\, g_2\,(x)\ d\omega_{\mathcal{F}^4(x_1,x_2,h,k)}\, dx, \tag{2.16}$$

Lemma 2.3. *Let $d \geq 6$. Then the partition of coordinates $z = (h,k)$, $y = (x_1,x_2)$ and $z = (x_1,h,k)$, $y = x_2$ are both admissible for the system $\mathcal{F}^4$.*

Proof. Consider a point $(h,k,x_1,x_2) \in S_{\mathcal{F}^4}$ at which the vectors h,k,x_1,x_2 are linearly independent. For the partition $z = (h,k)$, $y = (x_1,x_2)$, the base equation is $q(h,k) = 2h\cdot k$, and

$$\nabla_{(h,k)} q(h,k) = (2k, 2h) \neq 0.$$

The gradients of the remaining seven equations with respect to (x_1,x_2) are

$$(2x_1,0),\quad (2h,0),\quad (2k,0),$$
$$(2(x_1-x_2), 2(x_2-x_1)),$$
$$(0,2x_2),\quad (0,2(x_2+k)),\quad (0,2h).$$

The x_1-components of the first four vectors are linearly independent; after their coefficients vanish, the last three vectors are linearly independent because x_2,k,h are linearly independent. This proves admissibility of the first partition.

For the partition $z = (x_1,h,k)$, $y = x_2$, the base equations are

$$p_1(x_1),\quad p'_{12}(x_1,h),\quad p'_{12}(x_1,k),\quad q(h,k).$$

Their gradients with respect to (x_1, h, k) are

$$(2x_1, 0, 0), \quad (2h, 2x_1, 0), \quad (2k, 0, 2x_1), \quad (0, 2k, 2h),$$

and are linearly independent. The four gradients with respect to x_2 are

$$2x_2, \quad 2(x_2 + k), \quad 2(x_2 - x_1), \quad 2h.$$

Their span is the same as the span of x_1, x_2, h, k, so they are also linearly independent.

It remains to construct such a point. Set $x_1 = v_1$, $x_2 = v_2$, and let

$$S_1 := \{u \in \mathbb{R}^d : \ p_2(u) = 0, \ p_{12}(x_1, u) = 0\}.$$

As above, S_1 is a non-degenerate $(d-2)$-dimensional sphere. Choose $x_2 + h \in S_1$ so that $h \notin \mathrm{Span}\{x_1, x_2\}$. Then x_1, x_2, h are linearly independent. Consider

$$S_2 := S_1 \cap (x_2 + h^{\perp}) = \{u \in S_1 : \ (u - x_2) \cdot h = 0\}.$$

The hyperplane is not tangent to S_1 at x_2, since otherwise h would belong to $\mathrm{Span}\{x_1, x_2\}$. Hence S_2 is a non-degenerate sphere of dimension $d-3$, whose affine span has dimension $d-2$. For $d \geq 6$, this affine span cannot be contained in the three-dimensional space $\mathrm{Span}\{x_1, x_2, h\}$. We may therefore choose

$$x_2 + k \in S_2 \setminus \mathrm{Span}\{x_1, x_2, h\}.$$

Then $h \cdot k = 0$, the vectors h, k, x_1, x_2 are linearly independent, and $x_2, x_2 + h, x_2 + k$ all belong to S_1. Thus all eight equations in $\mathcal{F}^4$ hold. $\square$

Let $S_\perp := \{(h, k) \in \mathbb{R}^{2d} : \ h \cdot k = 0\}$ and let $\omega_\perp$ be the associated measure on it, defined in (4.3). For given h, k define the function

$$W(h,k) := \int \mathbf{1} \, d\omega_{\mathcal{F}^4_{h,k}}(x_1, x_2), \tag{2.17}$$

for $d\omega_\perp$-almost every (h, k). The equations force $|x_1|, |x_2|, |x_2 + h|, |x_2 + k| \lesssim 1$, and hence also $|h|, |k| \lesssim 1$. Thus, if desired, the constant function $\mathbf{1}$ in (2.17) may be replaced by a fixed smooth compactly supported cut-off in (x_1, x_2, h, k) which equals one on this set.

Then, by Lemma 2.3, one may write inequality (2.16) in the form

$$|\mathcal{N}_\lambda|^4 \lesssim \int \int \Delta_{\lambda bk, \lambda h} \, g_2 \, (x) \, W(h,k) \, d\omega_\perp(h,k) \, dx. \tag{2.18}$$

This is already a weighted localized geometric uniformity norm, however the weight function still depends on the configuration Γ_0. To remove this dependence we need some estimates on the weight function $W(h,k)$, summarized in the following.

Lemma 2.4. *Let $d \geq 6$. For $d\omega_\perp$-almost every (h,k), we have*

$$W(h,k) = C_{\Gamma_0, d} \, \frac{\left(\tau - |h|^2 - |k|^2\right)_+^{\frac{d-5}{2}}}{|h|^2 |k|^2} \; .$$

Moreover,

$$\int |W(h,k)| \, d\omega_\perp(h,k) \lesssim 1, \tag{2.19}$$

and

$$\int |\nabla W(h,k)| \, d\omega_\perp(h,k) \lesssim 1. \tag{2.20}$$

Also, let $0 < L \leq 1$ and let

$$\Omega_L := \{(h,k) : \ |h| \lesssim 1, \, |k| \lesssim 1, \, |h \cdot k| \lesssim L\} \subseteq \mathbb{R}^{2d}.$$

Then

$$\int_{\Omega_L} |W(h,k)|\, dh\, dk \lesssim L. \tag{2.21}$$

Proof. Let

$$s_{12} := \frac{t_1 + t_2 - t_{12}}{2} = v_1 \cdot v_2.$$

Fix nonzero h, k with $h \cdot k = 0$. In the first stage of the disintegration, x_1 satisfies

$$|x_1|^2 = t_1, \qquad x_1 \cdot h = 0, \qquad x_1 \cdot k = 0.$$

In other words, the base x_1-system is

$$\mathcal{B}_{h,k}(x_1) = \{|x_1|^2 - t_1,\ x_1 \cdot h,\ x_1 \cdot k\}.$$

The gradients $2x_1, h, k$ are linearly independent on its zero set, so this integration is admissible. Its zero set is a sphere of dimension $d-3$ and radius $t_1^{1/2}$. Its Leray density contains the inverse co-volume of the two linear constraints, and hence

$$\int \mathbf{1}\, d\omega_{\mathcal{B}_{h,k}}(x_1) = \frac{c_{d,t_1}}{|h||k|}.$$

For fixed h, k, x_1, denote the x_2-fiber system by $\mathcal{G}_{h,k,x_1}$. Its equations are

$$|x_2|^2 = t_2, \qquad x_1 \cdot x_2 = s_{12}, \qquad x_2 \cdot h = -\frac{|h|^2}{2}, \qquad x_2 \cdot k = -\frac{|k|^2}{2}.$$

Since x_1, h, k are pairwise orthogonal, this fiber is a sphere of dimension $d-4$ and radius

$$\begin{aligned} R(h,k) &= \left(t_2 - \frac{s_{12}^2}{t_1} - \frac{|h|^2}{4} - \frac{|k|^2}{4}\right)^{1/2} \\ &= \frac{1}{2}\big(\tau - |h|^2 - |k|^2\big)^{1/2}, \end{aligned}$$

where

$$\tau := 4\left(t_2 - \frac{s_{12}^2}{t_1}\right) = \frac{4(t_1 t_2 - s_{12}^2)}{t_1} > 0.$$

The Leray mass of this fiber is therefore

$$\int \mathbf{1}\, d\omega_{\mathcal{G}_{h,k,x_1}}(x_2) = c_{\Gamma_0,d} \frac{R(h,k)^{d-5}}{|h||k|}.$$

This gives above the explicit formula for the weight function $W(h,k)$.

The reduced base equation is $q(h,k) = 2h \cdot k$. Thus, if $d\omega_\perp$ is associated to $h \cdot k$, then the base Leray measure arising directly from $\mathcal{F}^4$ is $\frac{1}{2} d\omega_\perp$; the constant is absorbed into the implicit constant in (2.18). We now use the polar disintegration of the Leray measure on $S_\perp$. If

$$h = r\theta, \qquad k = s\phi, \qquad \theta \in S^{d-1}, \quad \phi \in S^{d-1} \cap \theta^\perp,$$

then

$$\int_{S_\perp} F(h,k)\, d\omega_\perp(h,k) = c_d \int_0^\infty \int_0^\infty r^{d-2} s^{d-2} \int_{S^{d-1}} \int_{S^{d-1}\cap\theta^\perp} F(r\theta, s\phi)\, d\sigma_\theta(\phi)\, d\sigma(\theta)\, dr\, ds.$$

where $c_d = 1$ when $d\sigma$ and $d\sigma_\theta$ denote the (un-normalised) induced surface measures (and otherwise c_d accounts for their normalisation). Write

$$\alpha := \frac{d-5}{2}, \qquad T := \tau - r^2 - s^2.$$

It follows that

$$\int_{S_\perp} |W|\, d\omega_\perp \lesssim \int_{\substack{r,s>0\\ r^2+s^2<\tau}} r^{d-4}s^{d-4}T^\alpha\, dr\, ds \lesssim 1,$$

which proves (2.19).

Away from the axes and the outer boundary, the radial extension satisfies

$$|\nabla W| \lesssim \frac{T^\alpha}{r^3s^2} + \frac{T^\alpha}{r^2s^3} + \frac{(r+s)T^{\alpha-1}}{r^2s^2}.$$

Therefore

$$\int_{S_\perp} |\nabla W|\, d\omega_\perp \lesssim \int_{\substack{r,s>0\\ r^2+s^2<\tau}} \big(r^{d-5}s^{d-4}T^\alpha + r^{d-4}s^{d-5}T^\alpha + (r+s)r^{d-4}s^{d-4}T^{\alpha-1}\big)\, dr\, ds.$$

The first two terms are integrable at the coordinate axes when $d > 4$, while the last term is integrable at $T = 0$ when $\alpha > 0$. Thus (2.20) holds for $d \geq 6$. On the full-$d\omega_\perp$-measure set where $h, k \neq 0$, $T \neq 0$, and the radial extension is differentiable, its ambient gradient is tangent to $S_\perp$, since it is of the form $(A(r,s)h, B(r,s)k)$ and hence is orthogonal to the normal vector (k,h).

Finally, ordinary polar coordinates in $\mathbb{R}^d \times \mathbb{R}^d$ and the elementary angular estimate

$$\int_{S^{d-1}}\int_{S^{d-1}} \mathbf{1}_{\{|\theta\cdot\phi|\lesssim L/(rs)\}}\, d\sigma(\phi)\, d\sigma(\theta) \lesssim \min\left(1, \frac{L}{rs}\right) \leq \frac{L}{rs}$$

give

$$\begin{aligned}\int_{\Omega_L} |W(h,k)|\, dh\, dk &\lesssim \int_{\substack{r,s>0\\ r^2+s^2<\tau}} r^{d-3}s^{d-3}T^\alpha \min\left(1, \frac{L}{rs}\right) dr\, ds\\ &\lesssim L \int_{\substack{r,s>0\\ r^2+s^2<\tau}} r^{d-4}s^{d-4}T^\alpha\, dr\, ds \lesssim L.\end{aligned}$$

This proves (2.21). □

2.3. **Local geometric uniformity norms.** The aim of this section is to show that the weighted expression in (2.18) can be majorized via a geometric analogue of Gowers U^2 uniformity norm defined in a smaller scale $L = \eta\lambda$, which is independent of the geometry of Γ_0.

Fix nonnegative functions $\rho, \chi \in C_c^\infty(\mathbb{R}^d)$ such that $\int \chi = 1$ and ρ is strictly positive on a sufficiently large neighborhood of the origin. We define,

Definition (Local geometric U^2 uniformity norm)**.**

$$\|f\|^4_{U^\perp(L)} = L^{d+2} \int\!\!\int \prod_{j=1}^4 f(x-y_j)\rho_L(y_j)\, d\omega_\square(y_1,y_2,y_3,y_4)\, dx, \tag{2.22}$$

where $d\omega_\square$ is associated to

$$y_1 + y_2 - y_3 - y_4 = 0, \qquad |y_1|^2 + |y_2|^2 - |y_3|^2 - |y_4|^2 = 0.$$

Let

$$\rho_L(y) := L^{-d}\rho(y/L), \qquad \chi_L(y) := L^{-d}\chi(y/L).$$

For a real-valued function f supported in a fixed bounded set, define

$$\mathcal{E}_L f(x,\xi,\alpha) := \int f(x-y)\rho_L(y)e^{2\pi i(\xi\cdot y+\alpha|y|^2)}\, dy.$$

The fourth-power integral below is nonnegative. Expanding the fourth power and using the oscillatory representation of the Leray measure (see section 4.3) gives the following equivalent representation of the local geometric uniformity norm

$$\|f\|_{U^\perp(L)}^4 := L^{d+2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}} |\mathcal{E}_L f(x,\xi,\alpha)|^4 \, d\alpha \, d\xi \, dx. \tag{2.23}$$

The factor L^{d+2} is the scale-invariant normalization, since $d\omega_\Box(Ly) = L^{3d-2} d\omega_\Box(y)$. Replacing ρ by another fixed bump function, or L by a fixed multiple of L, changes only the implicit constants below. We first show the following estimate for the singular weight that is needed in the smoothing argument.

Lemma 2.5. *Let $d \geq 14$. Uniformly for $|a|, |c| \lesssim 1$,*

$$\int_{S_\perp} |\nabla W(h+a, k+c)| \, d\omega_\perp(h,k) \lesssim 1. \tag{2.24}$$

Consequently, if $|t|, |s| \lesssim \eta \leq 1$, then

$$\int_{S_\perp} |W(h+t, k+s) - W(h,k)| \, d\omega_\perp(h,k) \lesssim \eta.$$

Proof. Write $H = h + a$ and $K = k + c$. The explicit formula in the proof of Lemma 2.4, and the fact that $(d-5)/2 > 1$, give on a fixed compact set

$$|\nabla W(H,K)| \lesssim |H|^{-3}|K|^{-2} + |H|^{-2}|K|^{-3} + |H|^{-2}|K|^{-2}.$$

It is therefore enough to observe that, for the pairs (p,q) above,

$$\sup_{|a|,|c| \lesssim 1} \int |H|^{-p}|K|^{-q} \delta\big((H-a)\cdot(K-c)\big) \, dH \, dK \lesssim 1, \tag{2.25}$$

where all variables are restricted to a fixed bounded set. Integrating first in K, the inner integral is over the affine hyperplane $c + (H-a)^\perp$ and is uniformly bounded when $q < d-1$. The remaining integral is bounded by

$$\int_{|H| \lesssim 1} |H|^{-p} |H-a|^{-1} \, dH \lesssim 1$$

uniformly in a, provided $p + 1 < d$. These conditions hold here which proves (2.24) and the last assertion of Lemma 2.5 follows from the fundamental theorem of calculus. □

For $0 < \eta \leq 1$ and $H, K \in \mathbb{R}^d$, let

$$M_\eta(H,K) := \int \chi_\eta(t)\chi_\eta(s) \delta\big((H-t)\cdot(K-s)\big) \, dt \, ds. \tag{2.26}$$

Thus $M_\eta(H,K)$ is the Leray mass of the portion of $S_\perp$ lying in an $O(\eta)$-neighborhood of (H,K).

Lemma 2.6. *Let $0 < \lambda \leq 1$, $0 < \eta \leq 1$, and $L = \eta\lambda$. Define*

$$\begin{aligned} I_{\lambda,\eta}(H,K) := \int & g(x) g\big(x + \lambda b(K-s)\big) g\big(x + \lambda(H-t)\big) \\ & \times g\big(x + \lambda(H-t) + \lambda b(K-s)\big) \chi_\eta(t)\chi_\eta(s) \\ & \times \delta\big((H-t)\cdot(K-s)\big) \, dt \, ds \, dx. \end{aligned}$$

If $|g| \leq 1$ and g is supported in a fixed bounded set, then

$$|I_{\lambda,\eta}(H,K)| \lesssim \min\left\{ M_\eta(H,K),\, \eta^{-2} \|g\|_{U^\perp(CL)}^4 \right\}, \tag{2.27}$$

with a constant $C = C_{\Gamma_0} > 0$.

Proof. The first bound is immediate from $|g| \le 1$. For the second, average the integral defining $I_{\lambda,\eta}$ over a further variable u with weight $\chi_\eta(u)$. Set

$$y_0 = u, \qquad y_1 = u + bs - bK,$$
$$y_2 = u + t - H, \qquad y_3 = u + t + bs - H - bK.$$

Then

$$y_0 + y_3 - y_1 - y_2 = 0$$

and, on the support of the delta function,

$$|y_0|^2 + |y_3|^2 - |y_1|^2 - |y_2|^2 = 2b(H-t)\cdot(K-s) = 0.$$

Moreover, after the change $X = x + \lambda u$, the four functions are $g(X - \lambda y_j)$, $0 \le j \le 3$. If

$$c_0 = 0, \qquad c_1 = -bK, \qquad c_2 = -H, \qquad c_3 = -H - bK,$$

then $|y_j - c_j| \lesssim \eta$ on the support of the three averaging functions. Choosing the bump function ρ in (2.22) sufficiently large, we have

$$\chi_\eta(u)\chi_\eta(t)\chi_\eta(s) \lesssim \eta^d \prod_{j=0}^{3} \rho_{C\eta}(y_j - c_j).$$

The preceding linear change of variables identifies $du\,dt\,ds\,\delta((H-t)\cdot(K-s))$, up to a nonzero constant depending on b, with δ representing the Leray measure. Letting $z_j = \lambda y_j$ and using homogeneity, we obtain

$$|I_{\lambda,\eta}(H,K)| \lesssim \eta^d \lambda^{d+2} \left| \int\!\!\int \prod_{j=0}^{3} g(X - z_j)\rho_{CL}(z_j - \lambda c_j)\, d\omega_\square(z)\, dX \right|.$$

For arbitrary centers a_j, using the oscillatory representation of $d\omega_\square$ and Holder's inequality, we get

$$\left| \int\!\!\int \prod_{j=0}^{3} g(X - z_j)\rho_{CL}(z_j - a_j)\, d\omega_\square(z)\, dX \right|$$
$$\lesssim (CL)^{-(d+2)} \|g\|^4_{U^\perp(CL)}.$$

Indeed, the four factors are paired in $L^4(dX\,d\xi\,d\alpha)$; translating a cutoff from the origin to a_j amounts to the measure-preserving change $(X,\xi,\alpha) \mapsto (X - a_j, \xi + 2\alpha a_j, \alpha)$. Since $L = \eta\lambda$, the prefactor is $O_{\Gamma_0}(\eta^{-2})$, proving (2.27). $\square$

The next estimate is where the singular factors $|H|^{-2}|K|^{-2}$ in W are used explicitly.

Lemma 2.7. *Let $0 < \eta \le 1$ and $0 \le \vartheta \le 1$. Then*

$$\int_{\{M_\eta > 0\}} W(H,K) M_\eta(H,K)^{1-\vartheta}\, dH\, dK \lesssim \eta^\vartheta. \tag{2.28}$$

Proof. Let $R = (|H|^2 + |K|^2)^{1/2}$. From (2.26) and the co-area formula,

$$M_\eta(H,K) \lesssim \frac{1}{\eta(\eta + R)}. \tag{2.29}$$

Indeed, when $R \lesssim \eta$ this follows by scaling and the local integrability of $|u|^{-1}$ in $\mathbb{R}^d$. When $R \gg \eta$, the gradient of $(H-t)\cdot(K-s)$ on the relevant portion of its zero set has size comparable to R, and the surface area of that portion is $O(\eta^{2d-1})$. Also, $M_\eta(H,K) > 0$ implies

$$|H \cdot K| \lesssim \eta(\eta + R). \tag{2.30}$$

Write $H = r\theta$, $K = s\phi$ and $T = \tau - r^2 - s^2$. By the explicit formula for W and an elementary angular estimate, the measure of the directions satisfying (2.30) is at most

$$C_d \min\left(1, \frac{\eta(\eta+R)}{rs}\right).$$

Hence, using (2.29) and then $\min(1, A) \le A$, the left side of (2.28) is at most a constant times

$$\int_{\substack{r,s>0\\ r^2+s^2<\tau}} r^{d-3}s^{d-3}T^{\frac{d-5}{2}} \frac{\min\left(1, \frac{\eta(\eta+R)}{rs}\right)}{[\eta(\eta+R)]^{1-\vartheta}}\, dr\, ds$$
$$\lesssim \eta^{\vartheta} \int_{\substack{r,s>0\\ r^2+s^2<\tau}} r^{d-4}s^{d-4}T^{\frac{d-5}{2}} (\eta+R)^{\vartheta}\, dr\, ds \lesssim \eta^{\vartheta}.$$

This proves the lemma. □

Let

$$T_\lambda(g) := \int\int \Delta_{\lambda bk,\lambda h} g(x) W(h,k)\, d\omega_\perp(h,k)\, dx,$$

so that (2.18) gives $|\mathcal{N}_\lambda|^4 \lesssim T_\lambda(g_2)$.

Proposition 2.1. *Let $0 < \lambda \le 1$, $0 < L \le c_{\Gamma_0}\lambda$, and put $\eta = L/\lambda$. For every $0 < \vartheta \le 1$ and every $g : Q_d \to [-1, 1]$, one has*

$$|T_\lambda(g)| \lesssim \eta^{-\vartheta} \|g\|_{U^\perp(CL)}^{4\vartheta} + \eta. \tag{2.31}$$

Proof. Averaging the weight in $T_\lambda(g)$ over translations t, s with density $\chi_\eta(t)\chi_\eta(s)$, Lemma 2.5 gives an error $O(\eta)$. After changing variable $H = h + t$ and $K = k + s$, the averaged expression is

$$\int W(H,K) I_{\lambda,\eta}(H,K)\, dH\, dK.$$

By Lemma 2.6 using $\min(A, B) \le A^{1-\vartheta}B^{\vartheta}$, we have

$$|I_{\lambda,\eta}(H,K)| \lesssim M_\eta(H,K)^{1-\vartheta} \left(\eta^{-2}\|g\|_{U^\perp(C_{\Gamma_0}L)}^4\right)^{\vartheta}.$$

Integrating and applying Lemma 2.7 proves (2.31). □

We can now state the main result of this section.

Theorem 2.1 (generalized von Neumann inequality)**.** *Let $d \ge 14$, $0 < \lambda \le 1$, and $0 < L \le c_{\Gamma_0}\lambda$. Let $\eta = L/\lambda$. If $g_i : Q_d \to [-1, 1]$, $0 \le i \le 3$, then for every $0 < \vartheta \le 1$,*

$$|\mathcal{N}_\lambda(g_0, g_1, g_2, g_3)| \lesssim \eta^{-\vartheta/4} \min_{0\le i\le 3} \|g_i\|_{U^\perp(C_{\Gamma_0}L)}^{\vartheta} + \eta^{1/4}. \tag{2.32}$$

In particular, taking $\vartheta = 1/4$,

$$|\mathcal{N}_\lambda(g_0, g_1, g_2, g_3)| \lesssim \left(\frac{\lambda}{L}\right)^{1/16} \min_{0\le i\le 3} \|g_i\|_{U^\perp(C_{\Gamma_0}L)}^{1/4} + \left(\frac{L}{\lambda}\right)^{1/4}. \tag{2.33}$$

Equivalently, for every $0 < \varepsilon < 1$,

$$|\mathcal{N}_\lambda(g_0, g_1, g_2, g_3)| \lesssim \varepsilon^{-1/4} \min_{0\le i\le 3} \|g_i\|_{U^\perp(C_{\Gamma_0}\varepsilon^4\lambda)}^{1/4} + O(\varepsilon). \tag{2.34}$$

Proof. The reduction leading to (2.18), followed by Proposition 2.1, gives (2.32) with g_2 in place of the minimum. As explained in the relabelling argument preceding (2.13), the vertices may be relabelled so that any prescribed g_i occupies this slot; the finitely many resulting constants are absorbed into C_{Γ_0}. Taking fourth roots gives (2.32). Formula (2.33) is the case $\vartheta = 1/4$, and (2.34) follows by taking L to be a sufficiently small fixed multiple of $\varepsilon^4\lambda$. □

3. From geometric $U^2_\perp$ to a local U^3 norm.

We now compare the standard geometric uniformity norm from (2.23)–(2.22) with an ordinary local U^3 norm. All functions in this subsection are extended by zero outside their original support, and the integration in the base variable x is over $\mathbb{R}^d$.

Choose an even nonnegative function $\zeta \in C_c^\infty(\mathbb{R}^d)$ which is strictly positive on a neighborhood of the origin, let

$$\vartheta = \zeta * \widetilde{\zeta}, \qquad \widetilde{\zeta}(y) = \zeta(-y), \qquad \vartheta_L(y) = L^{-d}\vartheta(y/L),$$

and define

$$\|f\|_{U^3(L)}^8 := \int \Delta_a\Delta_b\Delta_c f(x)\,\vartheta_L(a)\vartheta_L(b)\vartheta_L(c)\,dx\,da\,db\,dc. \tag{3.1}$$

This is nonnegative. Indeed, after writing each ϑ_L as an autocorrelation, the right side becomes the usual six-variable box average

$$\int \prod_{j=1}^{3} \zeta_L(p_j)\zeta_L(q_j) \prod_{\omega\in\{0,1\}^3} f\left(x + \sum_{j=1}^{3}\big((1-\omega_j)p_j + \omega_j q_j\big)\right) dx\,dp\,dq.$$

Thus (3.1) is a genuine local Gowers norm. Replacing ζ by another fixed bump function, or L by a fixed multiple of L, changes only the implicit constants below.

We first rewrite the geometric norm in "increment" coordinates. With the function ρ fixed in (2.23), define

$$\psi(h,k) = c_d \int \rho(z)\rho(z+h)\rho(z+k)\rho(z+h+k)\,dz, \tag{3.2}$$

where the constant $c_d > 0$ accounts for the normalization of the Leray measure. The change of variables

$$y_1 = z, \qquad y_2 = z+h+k, \qquad y_3 = z+h, \qquad y_4 = z+k$$

in (2.22) gives

$$\Lambda_{\perp,\lambda}(f) := \|f\|_{U^\perp(\lambda)}^4 = \lambda^{-(2d-2)} \int \Delta_h\Delta_k f(x)\,\psi(h/\lambda, k/\lambda)\,d\omega_\perp(h,k)\,dx. \tag{3.3}$$

Here we used

$$|y_1|^2 + |y_2|^2 - |y_3|^2 - |y_4|^2 = 2h\cdot k$$

and $d\omega_\perp(\lambda h, \lambda k) = \lambda^{2d-2} d\omega_\perp(h,k)$. The function ψ is nonnegative, smooth and compactly supported.

We next isolate the geometric estimates used in the Cauchy–Schwarz argument. For vectors $a_1, \dots, a_j$ write

$$\operatorname{Vol}(a_1,\dots,a_j) := \det\big((a_i\cdot a_l)_{i,l=1}^{j}\big)^{1/2}.$$

In particular, put

$$V(u_1,u_2) := \operatorname{Vol}(u_1,u_2) = |u_1 \wedge u_2|.$$

Lemma 3.1. *Assume $d \geq 5$. Let all parameters below be restricted by a fixed compact cut-off function. For*

$$F_{r,s}(u_1,u_2) := u_1\cdot s + u_2\cdot(r-u_1)$$

and $F_{r,s}(u_1,u_2) = 0$, let $\mathcal{W}(h,u_1,u_2;r,s)$ be the Leray mass in the k-variable of the three affine equations

$$\begin{aligned} h\cdot k &= 0,\\ u_1\cdot k &= -(h+u_1)\cdot(r-u_1),\\ u_2\cdot k &= -(h+u_2)\cdot s, \end{aligned} \tag{3.4}$$

with a fixed smooth compactly supported amplitude. Then, on the region

$$D(p) := vol(h, u_1, u_2) > 0 \qquad p = (h, u_1, u_2, r, s),$$

the function $\mathcal{W}$ is continuously differentiable and

$$\int_{\mathbb{R}^d} |\mathcal{W}(h, u_1, u_2; r, s)|\, dh \lesssim V(u_1, u_2)^{-1}, \tag{3.5}$$

$$\int_{\mathbb{R}^d} |\nabla_p \mathcal{W}(h, u_1, u_2; r, s)|\, dh \lesssim V(u_1, u_2)^{-2}, \tag{3.6}$$

where $\nabla_p \mathcal{W}$ denotes gradient whose values on the $\{D(p) = 0\}$ can be defined arbitrarily. $\{D(p) > 0\}$.

Proof. Let $p = (h, u_1, u_2, r, s)$ and define the $3 \times d$ matrix $B(p)$ and the vector $b(p)$,

$$B(p) = \begin{pmatrix} h^t \\ u_1^t \\ u_2^t \end{pmatrix}, \qquad b(p) = \begin{pmatrix} 0 \\ -(h + u_1) \cdot (r - u_1) \\ -(h + u_2) \cdot s \end{pmatrix}.$$

If $a(p, k)$ denotes the fixed smooth cut-off function. If $rank\,(B(p) = 3$, then

$$\mathcal{W}(p) = \int_{\mathbb{R}^d} a(p, k)\, \delta(B(p)k - b(p))\, dk. \tag{3.5}$$

Set

$$D = D(p) := \det(B(p)B(p)^t)^{1/2} = \mathrm{Vol}(h, u_1, u_2).$$

For $J \subseteq \{1, \dots, d\}$ $|J| = 3$, let $B_J = B_J(p)$ be the corresponding 3×3 minor. By the Cauchy–Binet identity,

$$D(p)^2 = \sum_{|J|=3} |\det B_J(p)|^2. \tag{3.6}$$

Choose a smooth homogeneous partition of unity $\{\chi_J\}_{|J|=3}$ subordinate to the sets

$$|\det B_J| \geq c_d D.$$

It may be chosen so that $|\nabla_p \chi_J| \lesssim D^{-1}$. Splitting (3.5) with this partition, write $k = (x, z)$, where $x = k_J \in \mathbb{R}^3$ and $z = k_{J^c} \in \mathbb{R}^{d-3}$. On the support of χ_J the equations determine

$$x = x_J(p, z) := B_J(p)^{-1}\big(b(p) - B_{J^c}(p)z\big),$$

and hence

$$\mathcal{W}_J(p) = \int_{\mathbb{R}^{d-3}} \frac{\chi_J(B(p))a(p, x_J(p, z), z)}{|\det B_J(p)|}\, dz. \tag{3.7}$$

The z-integration is over a fixed compact set. Since all matrix entries and parameters are bounded on the support of the amplitude, (3.6) and the support condition for χ_J give

$$\|B_J^{-1}\| \lesssim D^{-1}, \qquad |\nabla_p x_J| \lesssim D^{-1}, \qquad \big|\nabla_p |\det B_J|^{-1}\big| \lesssim D^{-2}.$$

Differentiating (3.7), including the derivative of χ_J, therefore

$$|\mathcal{W}(p)| \lesssim D(p)^{-1}, \qquad |\nabla_p \mathcal{W}(p)| \lesssim D(p)^{-2}. \tag{3.8}$$

Let $E = \mathrm{span}\{u_1, u_2\}$ and assume first that $V(u_1, u_2) > 0$. The wedge-product identity gives

$$D(h, u_1, u_2) = V(u_1, u_2)\,\mathrm{dist}(h, E).$$

Since h is restricted to a fixed compact set and the transverse dimension is $d - 2$, (3.8) implies, for $j = 1, 2$,

$$\int D(h, u_1, u_2)^{-j}\, dh \lesssim V(u_1, u_2)^{-j}, \tag{3.9}$$

provided $d-2>2$. This proves the Lemma. $\square$

Remark. Here $\nabla_p\mathcal{W}$ denotes the classical gradient on the open set $V(u_1,u_2)>0$. We note that this classical gradient represents the distributional gradient of $\mathcal{W}$.

Indeed, let

$$E=\operatorname{span}\{u_1,u_2\},\qquad \rho(h)=\operatorname{dist}(h,E).$$

Let $\chi_\delta=\chi(\rho/\delta)$, where χ vanishes on $[0,1]$ and equals one on $[2,\infty)$. For a smooth vector field compactly supported in the region $V>0$, integration by parts applied to $\chi_\delta\mathcal{W}$ produces the possible boundary term

$$\int \mathcal{W}\,\nabla_p\chi_\delta\cdot\Phi.$$

Since $|\mathcal{W}|\lesssim\rho^{-1}$ on the support of Φ and the normal space to $E=\{D(p)=0\}$ has dimension $d-2$, this term is $O_\Phi(\delta^{d-4})$, and therefore tends to zero when $d\geq 5$. Therefore,

$$-\int \mathcal{W}\,\operatorname{div}_p\Phi=\int\nabla_p\mathcal{W}\cdot\Phi.$$

Thus no additional singular part supported on $h\in E$ occurs in the distributional gradient $\mathcal{D}\mathcal{W}$ [**?**]

This point of view of using distributional gradients is helpful as the classical gradients of the kernels we deal with may not be defined on certain singular subvarieties originating from the Leray measure.

Let

$$\Phi(u,v)=u_1\cdot v_2+u_2\cdot v_1,\qquad \mathcal{S}=\{(u,v)\in\mathbb{R}^{4d}:\Phi(u,v)=0\},$$

and write $d\omega_{\mathcal{S}}$ for its Leray measure.

Lemma 3.2. *Let all four vector variables be restricted to a fixed compact set. If $d\geq 5$, then*

$$\int_{\mathcal{S}}\frac{d\omega_{\mathcal{S}}(u,v)}{V(u_1,u_2)}+\int_{\mathcal{S}}\frac{1+|v_1|^{-1}}{V(u_1,u_2)^2}\,d\omega_{\mathcal{S}}(u,v)\lesssim 1. \tag{3.10}$$

Proof. The integrand is defined only when $V=V(u_1,u_2)>0$. For fixed $u=(u_1,u_2)$, the v-fiber

$$M_u=\{v:u_1\cdot v_2+u_2\cdot v_1=0\}$$

is a hyperplane with normal (u_2,u_1). Hence

$$d\omega_{\mathcal{S}}(u,v)=\frac{du}{(|u_1|^2+|u_2|^2)^{1/2}}\,d\sigma_{M_u}(v). \tag{3.11}$$

Uniformly in $u\neq 0$, the compact hyperplane section has bounded mass, and

$$\int_{M_u}(1+|v_1|^{-1})\,d\sigma_{M_u}(v)\lesssim 1.$$

To see this, normalize $|u|=1$ and split according as $|u_1|\geq|u_2|$ or $|u_2|>|u_1|$; solve the hyperplane equation in one component of v_2 or v_1, respectively. The singular set $v_1=0$ has co-dimension at least $d-1$ inside M_u, so $|v_1|^{-1}$ is locally integrable for $d\geq 3$.

It remains to estimate integral in u. Write

$$u_2=t\theta,\qquad u_1=a\theta+y,\qquad y\in\theta^\perp.$$

Then $V(u_1,u_2)=t|y|$ and

$$du_1\,du_2=t^{d-1}\,dt\,d\theta\,da\,dy.$$

Consequently, the more singular integral in (3.10) is bounded by

$$\int\frac{t^{d-3}|y|^{-2}}{(t^2+a^2+|y|^2)^{1/2}}\,dt\,da\,dy\,d\theta,$$

which is locally finite for $d\geq 5$. The term with V^{-1} is similar and easier. $\square$

We can now identify the weighted U^3-form produced by the Cauchy–Schwarz inequality.

Proposition 3.1 (The weighted cube kernel)**.** *Let $d \geq 5$. There is a nonnegative compactly supported kernel $K \in BV(\mathbb{R}^{3d})$, depending only on d and on the fixed cutoff ρ in* (2.23)*, such that*

$$|\Lambda_{\perp,\lambda}(f)|^8 \lesssim \mathcal{Q}_{K,\lambda}(f), \tag{3.12}$$

where

$$\mathcal{Q}_{K,\lambda}(f) := \lambda^{-3d} \int K(r/\lambda, s/\lambda, w/\lambda)\Delta_w \Delta_r \Delta_s f(x)\, dx\, dw\, dr\, ds. \tag{3.13}$$

Moreover,

$$\|K\|_{L^1(\mathbb{R}^{3d})} + |\mathcal{D}K|(\mathbb{R}^{3d}) \lesssim 1, \tag{3.14}$$

where $\mathcal{D}K$ denotes the distributional gradient of the kernel K and $|\mathcal{D}K|(\mathbb{R}^{3d})$ its total variation.

Moreover, for each fixed (r, s), the function $w \mapsto K(r, s, w)$ is positive definite; in particular, $\mathcal{Q}_{K,\lambda}(f) \geq 0$.

Proof. We first assume $\lambda = 1$. Apply Cauchy–Schwarz twice to (3.3). U shifts (u_1, v_1) and (u_2, v_2) and, for $\epsilon = (\epsilon_1, \epsilon_2) \in \{0, 1\}^2$, put

$$q_\epsilon = (h + \epsilon_1 u_1 + \epsilon_2 u_2) \cdot (k + \epsilon_1 v_1 + \epsilon_2 v_2).$$

The four cutoff factors produced by the two squares are absorbed into a single nonnegative smooth compactly supported amplitude. The equations $q_\epsilon = 0$ are equivalent, by determinant-one row operations, to the three equations in (3.4) together with

$$u_1 \cdot v_2 + u_2 \cdot v_1 = 0.$$

Make the triangular change

$$r = u_1 + v_1, \qquad s = v_2, \qquad v_1 = r - u_1.$$

Expanding the two Cauchy–Schwarz inequalities and then shifting the base variable gives

$$\begin{aligned} |\Lambda_{\perp,1}(f)|^4 \lesssim \int & \mathcal{W}(h, u_1, u_2; r, s)\Delta_{s-u_2}\Delta_{r-u_1} f(x - h)\Delta_s \Delta_r f(x) \\ & \times dh\, dx\, d\omega_{r,s}(u_1, u_2)\, dr\, ds, \end{aligned} \tag{3.15}$$

where $d\omega_{r,s}$ is the Leray measure on $F_{r,s}(u_1, u_2) = 0$.

Let

$$A(h, u_1, u_2; r, s) = V(u_1, u_2)\mathcal{W}(h, u_1, u_2; r, s), \qquad d\mu_{r,s} = V(u_1, u_2)^{-1} d\omega_{r,s},$$

and set $G_{r,s} = \Delta_r \Delta_s f$. By Lemma 3.2, the total mass of $d\mu_{r,s}\, dr\, ds$ on the relevant compact set is bounded. After putting $y = x - h$, Cauchy–Schwarz in (y, u_1, u_2, r, s) yields

$$|\Lambda_{\perp,1}(f)|^8 \lesssim \int \left| \int A(h, u_1, u_2; r, s)G_{r,s}(y + h)\, dh \right|^2 dy\, d\mu_{r,s}\, dr\, ds. \tag{3.16}$$

Expanding the square, setting $w = h' - h$ and then $x = y + h$, gives (3.12), with

$$K(r, s, w) = \int_{F_{r,s}=0} \frac{1}{V(u_1, u_2)} \int A(h, u_1, u_2; r, s)A(h + w, u_1, u_2; r, s)\, dh\, d\omega_{r,s}(u_1, u_2). \tag{3.17}$$

Thus $K(r, s, \cdot)$ is an auto-correlation in the h-variable. Its Fourier transform in w is nonnegative, proving the final assertion of the proposition.

It remains to prove (3.14). Lemma 3.1 gives

$$\|A\|_{L^1_h} \lesssim 1, \qquad \|\nabla_{h,u_1,u_2,r,s} A\|_{L^1_h} \lesssim V^{-1}. \tag{3.18}$$

For fixed (u_1, u_2, r, s) put

$$B(u_1, u_2, r, s, w) = \int A(h)A(h + w)\, dh.$$

Young's inequality and (3.18) imply

$$\begin{aligned} &\|B\|_{L^1_w} \lesssim 1, \\ &\|\nabla_w B\|_{L^1_w} + \|\nabla_{u_1,u_2,r,s} B\|_{L^1_w} \lesssim V^{-1}. \end{aligned} \tag{3.19}$$

Consequently the density $C = V^{-1}B$ satisfies

$$\|C\|_{L^1_w} \lesssim V^{-1}, \qquad \|\nabla_{w,u_1,u_2,r,s} C\|_{L^1_w} \lesssim V^{-2}. \tag{3.20}$$

The L^1 bound for K and its variation in w now follow directly from Lemma 3.2.

To estimate the variation in r, s, return to the original compatibility variables $v_1 = r - u_1$ and $v_2 = s$. Away from $v_1 = 0$, the co-area formula for

$$(r - v_1) \cdot s + u_2 \cdot v_1 = 0$$

gives

$$K(r, s, w) = \int \frac{dv_1}{|v_1|} \int_{v_1^\perp} C\left(r - v_1, z - \frac{(r - v_1) \cdot s}{|v_1|^2} v_1, r, s, w\right) d\sigma_{v_1^\perp}(z). \tag{3.21}$$

Here the suppressed arguments of C include v_1 and $v_2 = s$. The point of (3.21) is that the integration domain in (v_1, z) is independent of r and s.

To make the limiting argument explicit, choose $\kappa \in C^\infty([0, \infty))$ with $0 \le \kappa \le 1$, $\kappa(t) = 0$ for $t \le 1$, and $\kappa(t) = 1$ for $t \ge 2$, and let K_δ be the right side of (3.21) with the additional factor $\kappa(|v_1|/\delta)$. For fixed $\delta > 0$ differentiation under the integral is legitimate. If

$$u_1 = r - v_1, \qquad u_2 = z - \frac{(r - v_1) \cdot s}{|v_1|^2} v_1,$$

then

$$|\nabla_r u_1| + |\nabla_s v_2| \lesssim 1, \qquad |\nabla_{r,s} u_2| \lesssim |v_1|^{-1}$$

on the fixed compact support. The chain rule and the co-area change back to $(u, v) \in \mathcal{S}$ therefore give

$$\begin{aligned} \|\nabla_{r,s} K_\delta\|_{L^1_{r,s,w}} &\lesssim \int_{\mathcal{S}} \big(1 + |v_1|^{-1}\big) \|\nabla_{u_1,u_2,r,s} C(u, v, \cdot)\|_{L^1_w}\, d\omega_{\mathcal{S}}(u, v) \\ &\lesssim \int_{\mathcal{S}} \frac{1 + |v_1|^{-1}}{V(u_1, u_2)^2}\, d\omega_{\mathcal{S}}(u, v) \lesssim 1. \end{aligned} \tag{3.22}$$

The same argument, using (3.20), gives uniformly in δ

$$\|K_\delta\|_1 + \|\nabla_w K_\delta\|_1 \lesssim 1. \tag{3.23}$$

Thus $K_\delta \in W^{1,1}(\mathbb{R}^{3d})$ and

$$\sup_{\delta > 0} \big(\|K_\delta\|_1 + |\mathcal{D} K_\delta|(\mathbb{R}^{3d})\big) \lesssim 1.$$

Moreover, by (3.20),

$$\begin{aligned} \|K - K_\delta\|_1 &\lesssim \int_{\mathcal{S} \cap \{|v_1| \le 2\delta\}} \|C(u, v, \cdot)\|_{L^1_w}\, d\omega_{\mathcal{S}}(u, v) \\ &\lesssim \int_{\mathcal{S} \cap \{|v_1| \le 2\delta\}} \frac{d\omega_{\mathcal{S}}(u, v)}{V(u_1, u_2)} \longrightarrow 0 \end{aligned} \tag{3.24}$$

by dominated convergence and Lemma 3.2. Lower semicontinuity of total variation under L^1 convergence cite/ implies

$$|\mathcal{D}K|(\mathbb{R}^{3d}) \le \liminf_{\delta\to 0} |DK_\delta|(\mathbb{R}^{3d}) \lesssim 1.$$

This proves $K \in BV$ and (3.14).

Finally, rescaling all increment variables by λ gives (3.13). The scaled kernel is

$$K_\lambda(r,s,w) = \lambda^{-3d} K(r/\lambda, s/\lambda, w/\lambda).$$

□

The last step is a general localization statement for an L^1 cube kernel. We include the quantitative version needed here.

Lemma 3.3 (Localization of a weighted cube form)**.** *Let $K \in BV(\mathbb{R}^{3d})$ be compactly supported, let $0 < L \le \lambda \le 1$, and let $K_\lambda(r,s,w) = \lambda^{-3d}K(r/\lambda, s/\lambda, w/\lambda)$. If $|f| \le 1$ and f is supported in a fixed bounded set, then*

$$\left| \int K_\lambda(r,s,w)\Delta_w\Delta_r\Delta_s f(x)\,dx\,dw\,dr\,ds \right| \lesssim \|K\|_1 \|f\|_{U^3(CL)} + \frac{L}{\lambda}|\mathcal{D}K|(\mathbb{R}^{3d}). \tag{3.25}$$

Proof. Let $P_L K_\lambda$ be the conditional expectation of K_λ on the product grid of cubes of side length L in the variables (r,s,w). Using L^1 Poincare inequality, or arguing directly, and scaling implies

$$\|K_\lambda - P_L K_\lambda\|_1 \lesssim L|DK_\lambda|(\mathbb{R}^{3d}) = \frac{L}{\lambda}|DK|(\mathbb{R}^{3d}). \tag{3.26}$$

Thus (3.26) bounds the error in (3.25)made by replacing K_λ by $P_L K_\lambda$.

It remains to handle one product fixed cube, $Q_r \times Q_s \times Q_w$. Write $r = r_0 + a$, $s = s_0 + b$, $w = w_0 + c$, where a, b, c range over cubes of side L. Expanding the three differences turns the corresponding integral into the local cube form of the eight functions

$$f_\omega = T_{\omega_1 r_0 + \omega_2 s_0 + \omega_3 w_0} f, \qquad \omega \in \{0,1\}^3.$$

After translating the three cubes to the origin, their normalized indicator functions are dominated, choosing the fixed constant C sufficiently large, by the kernels used in (3.1). Three applications of Cauchy–Schwarz therefore give the local Gowers inequality

$$\left| \int_{Q_r\times Q_s\times Q_w} \int \Delta_w\Delta_r\Delta_s f(x)\,dx\,dw\,dr\,ds \right| \lesssim L^{3d} \prod_{\omega\in\{0,1\}^3} \|f_\omega\|_{U^3(CL)}^{1/8}. \tag{3.27}$$

The norm in (3.1) is translation invariant, so the product on the right of (3.27) equals $\|f\|_{U^3(CL)}$. Multiplying by the constant value of $P_L K_\lambda$ on each cell and summing gives

$$|\mathcal{Q}_{P_L K_\lambda}(f)| \lesssim \|P_L K_\lambda\|_1 \|f\|_{U^3(CL)} \lesssim \|K\|_1 \|f\|_{U^3(CL)}.$$

Together with (3.26), this proves the lemma. □

Combining Proposition 3.1 and Lemma 3.3 yields the majorization of the local $U\perp^2$-norm via the local U^3-norm.

Theorem 3.1 (Geometric local $U^2_\perp$ versus local U^3)**.** *Let $d \ge 5$, $0 < \lambda \le 1$, $0 < \varepsilon \le 1$, and put $L = \varepsilon\lambda$. If $f : Q_d \to [-1,1]$, then*

$$\boxed{\|f\|_{U^\perp(\lambda)}^{32} \lesssim \|f\|_{U^3(CL)} + \varepsilon.} \tag{3.28}$$

Equivalently,

$$\Lambda_{\perp,\lambda}(f) \lesssim \|f\|_{U^3(C\varepsilon\lambda)}^{1/8} + O(\varepsilon^{1/8}). \tag{3.29}$$

The constants depend only on d and on the fixed cutoff functions.

Proof. Apply (3.12), followed by (3.25), with $L = \varepsilon\lambda$, and use (3.14). Since $\Lambda_{\perp,\lambda}(f) = \|f\|^4_{U^\perp(\lambda)}$, this gives (3.28); taking eighth roots gives (3.29). □

Combined with Theorem 2.1 this proves our main result Theorem 1.1.

Proof of Theorem 1.1. Use (2.32) with $\vartheta = 1$ and $L_1 \asymp \varepsilon^4\lambda$. This gives

$$|\mathcal{N}_\lambda| \lesssim \varepsilon^{-1} \min_i \|g_i\|_{U^\perp(C_{\Gamma_0}L_1)} + O(\varepsilon).$$

Apply Theorem 3.1 to the geometric norm at scale $C_{\Gamma_0}L_1$, with its small parameter chosen comparable to ε^{64}. The resulting additive term is $O(\varepsilon^2)$ before the factor ε^{-1}, and the final U^3 scale is a constant multiple of $\varepsilon^{68}\lambda$. This implies

$$|\mathcal{N}_\lambda(g_0, g_1, g_2, g_3)| \leq C\varepsilon^{-1} \min_{0\leq i\leq 3} \|g_i\|^{1/32}_{U^3(c\varepsilon^{68}\lambda)} + C\varepsilon,$$

Writing $\varepsilon := c\varepsilon^{68}$ Theorem 1.1 follows. □

4. Appendix: Measures on real algebraic sets.

We introduce a natural family of measures defined on real algebraic sets and discuss their their properties and representations that will be used throughout the paper.

4.1. **Basic definitions and properties.** Let $\mathcal{F} = \{f_1, \ldots, f_n\}$ be a family of polynomials $f_i : \mathbb{R}^d \to \mathbb{R}$. We will describe certain measures supported on the algebraic set

$$S_{\mathcal{F}} := \{x \in \mathbb{R}^d : \ f_1(x) = \ldots = f_n(x) = 0\}. \tag{4.1}$$

A point $x \in S_{\mathcal{F}}$ is called *non-singular* if the gradient vectors $\nabla f_1(x), \ldots, \nabla f_n(x)$ are linearly independent, and let $S^0_{\mathcal{F}}$ denote the set of non-singular points. It is easy to see, that if $S^0_{\mathcal{F}} \neq \emptyset$ then it is a relative open, dense subset of the corresponding component of $S_{\mathcal{F}}$. Moreover it is an $d-n$-dimensional sub-manifold of $\mathbb{R}^d$. If $x \in S^0_{\mathcal{F}}$ then there exists a set of coordinates, $J = \{j_1, \ldots, j_n\}$, with $1 \leq j_1 < \ldots < j_n \leq d$, such that

$$j_{\mathcal{F},J}(x) := det \left(\frac{\partial f_i}{\partial x_j}(x)\right)_{1\leq i\leq n, j\in J} \neq 0. \tag{4.2}$$

Accordingly, we will call a set of coordinates J *admissible*, if (4.2) holds for at least one point $x \in S^0_{\mathcal{F}}$, and will denote by $S_{\mathcal{F},J}$ the set of such points. It is clear that $S_{\mathcal{F},J}$ is a relative open and dense subset of $S_{\mathcal{F}}$ and is a also $d-n$-dimensional sub-manifold, moreover

$$S^0_{\mathcal{F}} = \bigcup_{J\ admissible} S_{\mathcal{F},J}.$$

We define a measure, near a point $x_0 \in S_{\mathcal{F},J}$ as follows. For simplicity of notation assume that $J = \{1, \ldots, n\}$ and let $\Phi(x) := (f_1, \ldots, f_n, x_{n+1}, \ldots, x_d)$. Then $\Phi : U \to V$ is a diffeomorphism on some open set $x_0 \in U \subseteq \mathbb{R}^d$ to its image $V = \Phi(U)$, moreover $S_{\mathcal{F}} = \Phi^{-1}(V \cap \mathbb{R}^{d-n})$. Indeed, $x \in S_{\mathcal{F}} \cap U$ if and only if $\Phi(x) = (0, \ldots, 0, x_{n+1}, \ldots, x_d) \in V$. Let $I = \{n+1, \ldots, d\}$ and write $x_I := (x_{n+1}, \ldots, x_d)$. Let $\Psi(x_I) = \Phi^{-1}(0, x_I)$ and in local coordinates x_I define the measure $\omega_{\mathcal{F}}$ via

$$\int g\, d\omega_{\mathcal{F}} := \int g(\Psi(x_I))\ |Jac^{-1}_\Phi(\Psi(x_I))|\, dx_I, \tag{4.3}$$

for a continuous function g supported on U. Note that $Jac_\Phi(x) = j_{\mathcal{F},J}(x)$, i.e. the Jacobian of the mapping Φ at $x \in U$ is equal to the expression given in (4.2), and that the measure $d\omega_{\mathcal{F}}$ is supported on $S_{\mathcal{F}}$. Define the local coordinates $y_j = f_j(x)$ for $1 \leq j \leq n$ and $y_j = x_j$ for $n < j \leq d$. Then

$$dy_1 \wedge \ldots \wedge dy_d = df_1 \wedge \ldots \wedge df_n \wedge dx_{n+1} \wedge \ldots \wedge dx_d = Jac_\Phi(x)\, dx_1 \wedge \ldots \wedge dx_d,$$

thus

$$dx_1 \wedge \ldots \wedge dx_d = Jac_\Phi(x)^{-1} df_1 \wedge \ldots \wedge df_n \wedge dx_{n+1} \wedge \ldots \wedge dx_d = df_1 \wedge \ldots \wedge df_n \wedge d\omega_{\mathcal{F}}.$$

This shows that the measure $d\omega_{\mathcal{F}}$ (given as a differential $d-n$-form on $S_{\mathcal{F}} \cap U$) is independent of the choice of local coordinates x_I. Then $\omega_{\mathcal{F}}$ is defined on $S^0_{\mathcal{F}}$ and moreover the set $S^0_{\mathcal{F}} \backslash S_{\mathcal{F},J}$ is of measure zero with respect to ω_F, as it is a proper analytic subset on $\mathbb{R}^{d-n}$ in any other admissible local coordinates.

A more geometric description of the measure $d\omega_{\mathcal{F}}$ can be given as follows. Let $x_0 \in S^*_{\mathcal{F}}$, and choose an orthonormal basis $e_1, \ldots, e_d$ of $\mathbb{R}^d$ such that $span\{e_1, \ldots, e_n\} = span\{\nabla f_1(x_0), \ldots, \nabla f_n(x_0)\}$ and $T_{x_0} S_{\mathcal{F}} = span\{e_{n+1}, \ldots, e_d\}$. This possible as $\nabla f_i(x_0)$ is orthogonal to the tangent space of $S_{\mathcal{F}}$ at x_0. Let $x_1, \ldots, x_d$ be the system of coordinates associated to the basis. Then the tangent of the mapping $\Phi(x)$ at x_0 is a block diagonal matrix and its Jacobian $Jac_\Phi(x_0)$ is the determinant of the $n \times n$ matrix whose rows are the gradient vectors $\nabla f_i(x_0)$. This further equals to the volume of the parallelepiped spanned by these vectors. Thus the density of $d\omega_{\mathcal{F}}$ at x_0 is $vol(\nabla f_1(x_0), \ldots, \nabla f_i(x_0))^{-1}$ with respect to the surface area measure, which is equal to $dx_{n+1} \ldots dx_d$ at x_0 in this coordinate system. This shows in particular that the measure $d\omega_F$ is invariant under rotations and translations.

Let $x = (y, z)$ be a partition of coordinates in $\mathbb{R}^d$, with $y = x_J$. For given z, write $f_{i,z}(y) := f_i(y, z)$ and $\mathcal{F}_z := \{f_{1,z}, \ldots, f_{n,z}\}$. Assume there is an admissible coordinate set $J_1 \subseteq J$, then locally, near a point $x_0 \in S_{\mathcal{F},J_1}$ we have

$$j_{\mathcal{F},J_1}(y,z) = det\left(\frac{\partial f_i}{\partial x_j}(y,z)\right)_{1 \leq i \leq n, j \in J_1} = det\left(\frac{\partial f_{i,z}}{\partial x_j}(y)\right)_{1 \leq i \leq n, j \in J_1} = j_{\mathcal{F}_z,J_1}(y) \neq 0. \tag{4.4}$$

In the corresponding local coordinates (y_1, z), with $y_1 = x_{J \backslash J_1}$, on $S_{\mathcal{F},J_1}$, we have

$$d\omega_{\mathcal{F}}(y_1, z) = |j^{-1}_{\mathcal{F},J_1}(y_1, z)|\, dy_1\, dz = |j^{-1}_{\mathcal{F}_z,J_1}(y_1)|\, dy_1\, dz = d\omega_{\mathcal{F}_z}(y_1)\, dz. \tag{4.5}$$

We say that a partition of the coordinates $x = (y, z)$ with $y = x_J$ is *admissible*, if there exists an admissible set of coordinates $J_1 \subseteq J$. In this case we have an analogue of Fubini's theorem for the measure $\omega_{\mathcal{F}}$, namely, for a continuous function $g \in C(\mathbb{R}^d)$,

$$\int g(x)\, d\omega_{\mathcal{F}}(x) = \int\int g(y,z)\, d\omega_{\mathcal{F}_z}(y)\, dz. \tag{4.6}$$

For a given set of coordinates x_J let $\nabla_{x_J} f(x) := (\partial_{x_j} f(x))_{j \in J}$. Then a partition of the coordinates $x = (y, z)$ with $y = x_J$ is admissible, if and only if the gradient vectors $\nabla_{x_J} f_1(x), \ldots, \nabla_{x_J} f_n(x)$, are linearly independent at at least one point $x \in S_{\mathcal{F}}$. Indeed in this case one may choose a set of coordinates $J_1 \subseteq J$ for which $j_{\mathcal{F},J_1}(x) \neq 0$.

Note that a partition $x = (y, z)$ cannot be admissible if some of the functions f_i depend only on the z-variables, as then $\partial f_i / \partial x_j = 0$ for all $j \in J$. In this case we have the following variant of (4.6). Suppose the functions f_i, depend only on $z = x_I$, i.e $f_i(y, z) = f_i(z)$ for $1 \leq i \leq m$ (after possibly re-indexing the functions f_i). At any point $x = (y, z) \in S^0_{\mathcal{F}}$ the gradients $\nabla_z f_1(x), \ldots, \nabla_z f_m(x)$

must be linearly independent, thus $m \leq |I|$. In this case, we say that the partition $x = (y, z)$ is *admissible* if there exists an admissible set of coordinates J so that $|J \cap I| = m$. Write $J_1 := J \cap I$ and $J_2 := J \backslash I$, $\mathcal{F}_1 := \{f_1, \ldots, f_m\}$ and $\mathcal{F}_2 := \{f_{m+1}, \ldots, f_n\}$. For this it is enough to know that the gradients $\nabla_y f_{m+1}(x), \ldots, \nabla_y f_n(x)$ are also linearly independent as then one find an admissible set of coordinates $J = J_1 \cup J_2$ with $J_2 \cap I = \emptyset$ and $|J_2| = n - m$. It follows that

$$j_{\mathcal{F},J}(y,z) = j_{\mathcal{F}_1,J_1}(z)\, j_{\mathcal{F}_2,J_2}(y,z),$$

moreover $j_{\mathcal{F}_2,J_2}(y,z) = j_{\mathcal{F}_{2,z},J_2}(y)$ as it only involves partial derivatives with respect to the y-variables. Thus we have

$$\int g(x)\, d\omega_{\mathcal{F}}(x) \;=\; \int\int g(y,z)\, d\omega_{\mathcal{F}_{2,z}}(y)\, d\omega_{\mathcal{F}_1}(z). \tag{4.7}$$

We will use (4.7) together with the Cauchy-Schwarz inequality to estimate certain multilinear expressions, similarly as done in the discrete case [5]. Suppose $g(y,z) = g_1(z)g_2(y)$ with $|g_i| \leq 1$ and is compactly supported, then one has

$$\left|\int\int g_1(z)g_2(y)\, d\omega_{\mathcal{F}_{2,z}}(y)\, d\omega_{\mathcal{F}_1}(z)\right|^2 \lesssim \int\int\int g_2(y)g_2(y')\, d\omega_{\mathcal{F}_{2,z}}(y)\, d\omega_{\mathcal{F}_{2,z}}(y')\, d\omega_{\mathcal{F}_1}(z). \tag{4.8}$$

The above integral on the right side may be written as

$$I = \int g_2(y)g_2(y')\, d\omega_{\mathcal{F}^2}(y,y',z), \tag{4.9}$$

where $\mathcal{F}^2(y,y',z) = \{\mathcal{F}_1(z), \mathcal{F}_2(y,z)), \mathcal{F}_2(y',z)\}$, as the partition $\underline{y} = (y,y')$, $\underline{z} = z$ is admissible for the system $\mathcal{F}^2$. By a change of variables $y' = y + h$ and introducing the notations $\Delta_h g\,(y) := g(y)g(y+h)$ and $\partial_h \mathcal{F}_2(y,z) := \mathcal{F}_2(y+h,z) - \mathcal{F}_2(y,z)$, we have

$$I = \int \Delta_h g\,(y)\, d\omega_{\mathcal{F}^2}(y,h,z), \tag{4.10}$$

where $\mathcal{F}^2(y,h,z) = \{\mathcal{F}_1(z), \mathcal{F}_2(y,z), \partial_h \mathcal{F}_2(y,z)\}$.

Here we used fact that if a system of polynomials $\mathcal{F}' = \{g_1, \ldots, g_n\}$ is obtained from the system $\mathcal{F} = \{f_1, \ldots, f_n\}$ by a simple linear transformation, i.e. if $g_i = \sum_{j=1}^n a_{ij} f_j$ where $A = \{a_{ij}\}$ is a non-singular matrix, then clearly $S_{\mathcal{F}'} = S_{\mathcal{F}}$ and $dg_1 \wedge \ldots \wedge dg_n = \det(A)\, df_1 \wedge \ldots \wedge df_n$, thus $d\omega_{\mathcal{F}'} = |\det(A)|^{-1} d\omega_{\mathcal{F}}$.

4.2. **Linear and quadratic maps.** We'll be mostly dealing with systems $\mathcal{F}$ where the polynomials f_i are either linear or quadratic. It is not strictly necessary to include linear forms but will make some our calculations more transparent. Consider the subspace $S_{\mathcal{F}} \subseteq \mathbb{R}^{kd}$ where $\mathcal{F}$ represents the linear map

$$\mathfrak{f}(x_1, \ldots, x_k) = a_1 x_1 + \ldots + a_{k-1}x_{k-1} - x_k$$

with $x_i = (x_{ij})_{j=1,\ldots,d} \in \mathbb{R}^d$ and $a_i \in \mathbb{R}$ for $i = 1, \ldots, k$. In other words $\mathcal{F} = \{l_1, \ldots, l_d\}$ with $l_j(x_1, \ldots, x_d) = \sum_{i=1}^{k-1} a_i x_{ij} - x_{kj}$. In this case $J = \{k1, \ldots, kd\}$ is an admissible system of coordinates and clearly $j_{\mathcal{F},J}(x) \equiv 1$, thus we have

$$\int g(x_1, \ldots, x_k)\, d\omega_{\mathcal{F}} = \int g(x_1, \ldots, x_{k-1}, a_1x_1 + \ldots + a_{k-1}x_{k-1})\, dx_1 \ldots dx_{k-1}. \tag{4.11}$$

More generally, consider the system $\mathcal{F} = \{\mathfrak{f}, f_1, \ldots, f_n\}$ with $\mathfrak{f} = (l_1, \ldots, l_d)$ being the linear map given above. Let $x' = (x_1, \ldots, x_{k-1}) \in \mathbb{R}^{(k-1)d}$, $L(x') := a_1x_1 + \ldots + a_{k-1}x_{k-1}$ and define $f_i'(x') := f_i(x', L(x'))$, for $i = 1, \ldots, n$. Writing $\mathcal{F}' = \{f_1', \ldots, f_n'\}$, we have that $x' \in S_{\mathcal{F}'}$ if and only if $x = (x', L(x')) \in S_{\mathcal{F}}$. Moreover if J' is an admissible set of coordinates for $\mathcal{F}'$ then it is easy to see that $J := J' \cup \{k1, \ldots, kd\}$ is an admissible set of coordinates for $\mathcal{F}$, and $j_{\mathcal{F},J}(x) = j_{\mathcal{F}',J'}(x')$ when $x = (x', L(x'))$. Thus, we have

$$\int g(x_1, \ldots, x_k)\, d\omega_{\mathcal{F}}(x_1, \ldots, x_k) = \int g(x_1, \ldots, x_{k-1}, a_1x_1 + \ldots + a_{k-1}x_{k-1})\, d\omega_{\mathcal{F}'}(x_1, \ldots, x_{k-1}). \tag{4.12}$$

We will also often deal with algebraic sets given as the intersection of spheres. Let $x_1, \ldots, x_n \in \mathbb{R}^d$, $r_1, \ldots, r_n > 0$ and $\mathcal{F} = \{f_1, \ldots, f_n\}$ where $f_i(x) = |x - x_i|^2 - r_i^2$ for $i = 1, \ldots, n$. Then $S_{\mathcal{F}}$ is the intersection of spheres centered at the points x_i of radius r_i. If the points $x_1, \ldots, x_n$ are in general position (i.e they span an $n-1$-dimensional affine subspace), then a point $x \in S_{\mathcal{F}}$ is non-singular if $x \notin \mathit{span}\{x_1, \ldots, x_n\}$, i.e if x cannot be written as an affine combination of $x_1, \ldots, x_n$. Indeed, since $\nabla f_i(x) = 2(x - x_i)$ we have that

$$\sum_{i=1}^n a_i \nabla f_i(x) = 0 \iff \sum_{i=1}^n a_i\, x = \sum_{i=1}^n a_i x_i,$$

which implies $\sum_{i=1}^n a_i = 0$ and $\sum_{i=1}^n a_i x_i = 0$. By replacing the equations $|x - x_i|^2 = r_i^2$ with $|x - x_1|^2 - |x - x_i|^2 = r_1^2 - r_i^2$, which is of the form $x \cdot (x_1 - x_i) = c_i$, for $i = 2, \ldots, n$, it follows that $S_{\mathcal{F}}$ is the intersection of sphere with an $n-1$-codimensional affine subspace, perpendicular to the affine subspace spanned by the points x_i. Thus $S_{\mathcal{F}} = \emptyset$, $S_{\mathcal{F}} = \{p\}$, or $S_{\mathcal{F}}$ is an n-codimensional smooth submanifold of $\mathbb{R}^d$. Thus if $S_{\mathcal{F}}$ has one point $x \notin \mathit{span}\{x_1, \ldots, x_n\}$ then all of its points are non-singular, i.e. $S_{\mathcal{F}} = S^0_{\mathcal{F}}$.

4.3. **Oscillatory integral representations.** Formally one may write

$$d\omega_{\mathcal{F}} = \delta_{\{f_1 = \ldots = f_n = 0\}} = \int_{\mathbb{R}^n} e^{2\pi i(\alpha_1 f_1 + \ldots + \alpha_n f_n)}\, d\alpha_1 \ldots d\alpha_n. \tag{4.13}$$

To make sense of the above, let $0 \le \psi(\alpha) \le 1$ be a smooth cut-off which is identically 1 near the origin, and let $\psi_\varepsilon(\alpha) = \psi(\varepsilon\alpha)$ with $\alpha = (\alpha_1, \ldots, \alpha_n)$. Define

$$I_{\mathcal{F},\varepsilon}(x) := \int_{\mathbb{R}^n} e^{2\pi i(\alpha_1 f_1 + \ldots + \alpha_n f_n)(x)} \psi(\varepsilon\alpha)\, d\alpha_1 \ldots d\alpha_n. \tag{4.14}$$

We will show that for any test function $g \in C_0(\mathbb{R}^d)$,

$$\int g(x)\, I_{\mathcal{F},\varepsilon}(x)\, dx \ \to \ \int g(x)\, d\omega_{\mathcal{F}}(x), \qquad as \ \ \varepsilon \to 0, \tag{4.15}$$

and interpret the right side of (4.13) as the limit of the integrals $I_{\mathcal{F},\varepsilon}(x)$ as $\varepsilon \to 0$. To show the validity of formula (4.14) we use local coordinates on $S^0_{\mathcal{F}}$. By a partition of unity assume that $\mathit{supp}\ g \subseteq U$, where U is a local coordinate chart, so that there exists a diffeomorphism $\Phi : U \to V$ of the form $\Phi(x) = y$ with $y_i = f_i(x)$ for $1 \le i \le n$ and $y_i = x_i$ for $n < i \le d$. Let $\Psi = \Phi^{-1}$. Then $dy = j_{\mathcal{F},J}(x) dx$, with $J = \{1, \ldots, n\}$, thus

$$\int_{\mathbb{R}^d} e^{2\pi i(\alpha_1 f_1 + \ldots + \alpha_n f_n)(x)} g(x)\, dx = \int_{\mathbb{R}^d} e^{2\pi i(\alpha_1 y_1 + \ldots + \alpha_n y_n)} g(\Psi(y))\, j^{-1}_{\mathcal{F},J}(\Psi(y))\, dy,$$

hence

$$\int_{\mathbb{R}^d} g(x)\, I_{\mathcal{F},\varepsilon}(x)\, dx = \int_{\mathbb{R}^d} g(\Psi(y))\, j^{-1}_{\mathcal{F},J}(\Psi(y))\, \widehat{\psi}_\varepsilon(y_1, \ldots, y_n)\, dy.$$

Letting $\varepsilon \to 0$, we have that

$$\int_{\mathbb{R}^d} g(x)\, I_{\mathcal{F},\varepsilon}(x)\, dx \ \to \ \int_{\mathbb{R}^{d-n}} g(\Psi(0, y'))\, j^{-1}_{\mathcal{F},J}(\Psi(0, y'))\, dy' = \int g(x)\, d\omega_{\mathcal{F}}(x),$$

using the notation $y' = (y_{n+1}, \ldots, y_d)$.

4.4. **Homogeneity.** We say that a polynomial f_i is a form of degree k_i if $f_i(\lambda x) = \lambda^{k_i} f_i(x)$ and call $k = k_1 + \ldots + k_n$ the total degree of a system of forms $\mathcal{F} = \{f_1, \ldots, f_n\}$. Note that for any admissible coordinate set J, we have that $j_{\mathcal{F},J}(\lambda y) = \lambda^{k-n}\, j_{\mathcal{F},J}(y) \neq 0$, thus by (4.3)

$$d\omega_{\mathcal{F}}(\lambda y) = \lambda^{n-k} \lambda^{d-n}\, d\omega_{\mathcal{F}}(y) = \lambda^{d-k}\, d\omega_{\mathcal{F}}(y). \tag{4.16}$$

Often we'll be dealing with bounded sets $S_{\mathcal{F},t} = \{x \in \mathbb{R}^d :\ \mathcal{F}(x) = t\}$ for some $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ and $\mathcal{F}$ being a system of forms as above. Then $\lambda S_{\mathcal{F},t} = S_{\mathcal{F},\lambda\circ t}$, where $\lambda S_{\mathcal{F},t}$ denotes the dilate of the set $S_{\mathcal{F},t}$ by a factor of $\lambda > 0$ and $\lambda \circ t = (\lambda^{k_1} t_1, \ldots, \lambda^{k_n} t_n)$. Then by (4.16)

$$\int_{\lambda S_{\mathcal{F},t}} \phi(x)\, d\omega_{\mathcal{F}}(x) = \int_{S_{\mathcal{F},t}} \phi(\lambda y)\, d\omega_{\mathcal{F}}(\lambda y) = \lambda^{d-k} \int_{S_{\mathcal{F},t}} \phi(\lambda y)\, d\omega_{\mathcal{F}}(y). \tag{4.17}$$

In particular choosing $\phi \equiv 1$ on $S_{\mathcal{F},t}$ we have that

$$\omega_{\mathcal{F}}(\lambda S_{\mathcal{F},t}) = \lambda^{d-k}\, \omega_{\mathcal{F}}(S_{\mathcal{F},t}). \tag{4.18}$$

We'll normalize the integrals on the left side of (4.17) to ensure that they are bounded independent of λ.

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